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具有Lévy跳跃驱动的Rosenzweig-MacArthur捕食者–食饵模型的渐近行为

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摘 要

本文主要利用随机分析等相关知识，根据进化稳定策略框架对Rosenzweig-MacArthur捕食者–食饵模型进行改进，证明了具有Lévy跳跃驱动的Rosenzweig-MacArthur捕食者–食饵模型解的存在唯一性和随机有界性，并探究了该模型的灭绝性、持久性和平稳分布。

关键词

Rosenzweig-MacArthur捕食者–食饵模型，Lévy跳跃，灭绝性，平均持久性，平稳分布

Asymptotic Behavior of Rosenzweig-MacArthur Predator-Prey Model with Lévy Jump Drive

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Abstract

In this paper, we use stochastic analysis and other related knowledge to improve the Rosenzweig-MacArthur predator-prey model according to the evolutionary stable strategy framework, prove the existence, uniqueness and random boundedness of the solution of the Rosenzweig-MacArthur predator-prey model driven by Lévy jump, and explore the extinction, persistence and stationary

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distribution of the model.

Keywords

Rosenzweig-MacArthur Predator-Prey Model, Lévy Jumps, Extinction, Average Persistence, Smooth Distribution

1. 引言

众所周知，1956 年 Lotka 和 Volterra 提出了经典的 Lotka-Volterra 模型，用于描述捕食者和食饵种群的动态发展[1] [2] [3] [4]，但是 Lotka-Volterra 模型并没有考虑到现实生态环境的复杂性和变化性，无法那么贴切的描述真实情况。在 1963 年 M. Rosenzweig 和 R. MacArthur 提出的 Rosenzweig-MacArthur 捕食者－食饵模型考虑了更多现实影响因素，对 Lotka-Volterra 模型进行了扩展和改进，使模型更贴近实际情况[5]，研究了生态稳定以及捕食者与食饵种群之间的相互影响和动态进化发展，吸引了许多研究者的注意[6]-[11]。

Rosenzweig-MacArthur 的猎物－捕食者系统建立在 Lotka-Volterra 模型的基础上，考虑了更多的生态因素，使得模型更适用于描述复杂的生态系统动态。让 x_1 和 y_1 如分别表示猎物和捕食者的种群大小。假设系统的动态遵循逻辑－高斯型模型：

$$\begin{cases} \dfrac{dx_1}{dt} = x_1\left(f_1(x_1) - y_1\phi_{11}(x_1)\right), \\ \dfrac{dx_2}{dt} = y_1\left[-c_1 + kx_1\phi_{11}(x_1)\right]. \end{cases} \tag{1.1}$$

在这里，$x_1 f_1(x_1)$ 为猎物的生长率，$x_1 y_1 \phi_{11}(x_1)$ 为捕食率，$c_1 y_1$ 为捕食者的死亡率，且 $c_1 > 0$，其中：

$$\phi_{11}(x) = \frac{a}{b_{11} + x}$$

$$f_1(x) = r_1\left(1 - \frac{x}{K}\right)$$

a 和 b_{11} 为正常数，$r_1 > 0$ 为速率常数和 $K > 0$ 为猎物种群的承载能力。

考虑到生物多样性，在自然环境中，可能会出现突变体种群，并且突变体种群很可能拥有与原始种群不同的生存率和死亡率，对其他种群造成或大或小的影响。为了研究更适合现实情况的模型，2021 年，Grunert 等人[12]根据进化稳定策略框架对 Rosenzweig-MacArthur 捕食者－食饵模型进行改进，得到新的模型：

$$\begin{cases} dx_1 = x_1\left(f_1(x_1+x_2) - y_1\varphi_{11}(x_1+x_2) - y_2\varphi_{12}(x_1+x_2)\right)dt, \\ dy_1 = y_1\left[-c_1 + kx_1\varphi_{11}(x_1+x_2) + kx_2\varphi_{21}(x_1+x_2)\right]dt, \\ dx_2 = x_2\left(f_2(x_1+x_2) - y_1\varphi_{21}(x_1+x_2) - y_2\varphi_{22}(x_1+x_2)\right)dt, \\ dy_2 = y_2\left[-c_2 + kx_1\varphi_{12}(x_1+x_2) + kx_2\varphi_{22}(x_1+x_2)\right]dt. \end{cases} \tag{1.2}$$

其中 x_1 和 y_1 如分别表示原始的食饵种群和捕食者种群；x_2 和 y_2 分别表示相应的突变种群。$f_i(x)=r_i\left(1-\frac{x}{K_i}\right)$ 和 $\varphi_{ij}(x)=\frac{a}{b_{ij}+x}(i,j=1,2)$，$a$，$k$ 和 c，$c_i,r_i,K_i,c_{ij}(i=1,2)$ 都是正常数。

然而，生物种群不可避免会遭受严重的环境扰动，如海、火山、鸟类流感、SARS、洪水、飓风、地震、有毒污染物等，这些现象不能用随机性来描述连续模型。因此，将跳跃过程引入到种群系统是可行的，吸引了许多研究者的注意[13]-[22]。在本文中，我们将 Lévy 跳跃引入到 Rosenzweig-MacArthur 捕食者－食饵模型：

$$\begin{cases}\mathrm{d}x_1=\left\{x_1\left[f_1(x_1+x_2)-y_1\varphi_{11}(x_1+x_2)-y_2\varphi_{12}(x_1+x_2)\right]\right\}\mathrm{d}t\\ \qquad +\sigma_1x_1(t)\mathrm{d}w(t)+\int_z\psi_1(z)x_1(t)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\\ \mathrm{d}y_1=\left\{y_1\left[-c_1+kx_1\varphi_{11}(x_1+x_2)+kx_2\varphi_{21}(x_1+x_2)\right]\right\}\mathrm{d}t\\ \qquad +\sigma_2y_1(t)\mathrm{d}w(t)+\int_z\psi_2(z)y_1(t)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\\ \mathrm{d}x_2=\left\{x_2\left[f_2(x_1+x_2)-y_1\varphi_{21}(x_1+x_2)-y_2\varphi_{22}(x_1+x_2)\right]\right\}\mathrm{d}t\\ \qquad +\sigma_3x_2(t)\mathrm{d}w(t)+\int_z\psi_3(z)x_2(t)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\\ \mathrm{d}y_2=\left\{y_2\left[-c_2+kx_1\varphi_{12}(x_1+x_2)+kx_2\varphi_{22}(x_1+x_2)\right]\right\}\mathrm{d}t\\ \qquad +\sigma_4y_2(t)\mathrm{d}w(t)+\int_z\psi_4(z)y_2(t)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\end{cases} \tag{1.3}$$

其中 $\sigma_i(i=1,2)$ 是噪声强度，$w_i(t)(i=1,2)$ 是定义在完全概率空间 $(\Omega,\mathcal{F},P)$ 上的标准维纳过程，$\psi_i(i=1,2)$ 为有界函数。N 是在可测子集 $Z\subseteq[0,+\infty)$ 上具有特征测度入的泊松计数测度，其中 $\lambda(Z)<+\infty$，且 $\tilde{N}_i(\mathrm{d}t,\mathrm{d}z)=N(\mathrm{d}t,\mathrm{d}z)-\lambda(\mathrm{d}z)\mathrm{d}t$。

本文的的研究目的是研究具有 Lévy 驱动的 Rosenzweig-MacArthur 捕食者－猎物模型的动力学行为。通过加入 Lévy 驱动，模拟真实生态环境，探究 Lévy 驱动对捕食者和猎物和群的动态影响。因为在现实中的环境扰动往往是比较复杂的，Lévy 驱动的非线性和跳跃性质可以更好的贴近真实的捕食者－猎物种群动态过程，提高了模型对生态系统的拟合度。同时，还能为生态学研究提供新的思路，具有一定的参考意义。

2. 随机有界性

本小节，主要考虑具有 Lévy 跳跃驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)的正解和有界性。

定理 2.1 对于任意给定的初始条件 $(x_1(0),x_2(0),y_1(0),y_2(0))$，则具有 Lévy 跳跃驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)存在唯一的全局正解 $(x_1(t),x_2(t),y_1(t),y_2(t))\in\mathbb{R}_+^4$，$t\in(0,\tau_e)$，其中 τ_e 是爆炸时间。

证明：令

$$\overline{x_1}(t)=\ln x_1(t),\overline{x_2}(t)=\ln x_2(t),\overline{y_1}(t)=\ln y_1(t),\overline{y_2}(t)=\ln y_2(t),$$

运用 Itô 公式，得到：

$$
\begin{cases}
\mathrm{d}\overline{x_1}(t)=\left\{f_1\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)-\mathrm{e}^{y_1(t)}\varphi_{11}\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)+\varphi_{12}\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)-\dfrac{\sigma_1^2}{2}\right\}\mathrm{d}t \\
\qquad +\int_z\left[\ln\left(1+\psi_1(z)\right)-\psi_1(z)\right]v(\mathrm{d}z)\mathrm{d}t+\sigma_1\mathrm{d}w(t)+\int_z\left(1+\psi_1(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \\
\mathrm{d}\overline{y_1}(t)=\left\{-c_1+k\mathrm{e}^{x_1(t)}\varphi_{11}\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)+k\mathrm{e}^{x_2(t)}\varphi_{21}\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)-\dfrac{\sigma_2^2}{2}\right\}\mathrm{d}t \\
\qquad +\int_z\left[\ln\left(1+\psi_2(z)\right)-\psi_2(z)\right]v(\mathrm{d}z)\mathrm{d}t+\sigma_2\mathrm{d}w(t)+\int_z\left(1+\psi_2(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \\
\mathrm{d}\overline{x_2}(t)=\left\{f_2\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)-\mathrm{e}^{y_1(t)}\varphi_{21}\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)+\varphi_{22}\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)-\dfrac{\sigma_3^2}{2}\right\}\mathrm{d}t \\
\qquad +\int_z\left[\ln\left(1+\psi_3(z)\right)-\psi_3(z)\right]v(\mathrm{d}z)\mathrm{d}t+\sigma_3\mathrm{d}w(t)+\int_z\left(1+\psi_3(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \\
\mathrm{d}\overline{y_2}(t)=\left\{-c_2+k\mathrm{e}^{x_1(t)}\varphi_{12}\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)+k\mathrm{e}^{x_2(t)}\varphi_{22}\left(\mathrm{e}^{x_1(t)}+\mathrm{e}^{x_2(t)}\right)-\dfrac{\sigma_4^2}{2}\right\}\mathrm{d}t \\
\qquad +\int_z\left[\ln\left(1+\psi_4(z)\right)-\psi_4(z)\right]v(\mathrm{d}z)\mathrm{d}t+\sigma_4\mathrm{d}w(t)+\int_z\left(1+\psi_4(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z).
\end{cases}
\tag{2.1}
$$

显然上面式子(2.1)的系数是满足局部 Lipschitz 条件的，故对于任意给定的初始条件 $\left(x_1(0),x_2(0),y_1(0),y_2(0)\right)$，对于 $t\in(0,\tau_e)$，存在唯一的局部解 $\left(x_1(t),x_2(t),y_1(t),y_2(t)\right)$，将局部解推广到全局解，只需证明 $\tau_e=\infty$ 几乎处处存在即可。

令 $n_0>0$ 充分大，使得解 $\left(x_1(t),x_2(t),y_1(t),y_2(t)\right)$ 在区间 $\left[\dfrac{1}{n_0},kn_0\right]$ 中。对每个 $n>n_0$，定义停止时间为

$$
\tau_n=\inf\left\{t\in[0,\tau_e):x_1(t)\notin\left(\frac{1}{n_0},n_0\right)\text{ or }y_1(t)\notin\left(\frac{1}{n_0},n_0\right)\text{ or }x_2(t)\notin\left(\frac{1}{n_0},n_0\right)\text{ or }y_2(t)\notin\left(\frac{1}{n_0},n_0\right)\right\}
$$

显然，当 $n\to\infty$ 时，τ_n 是单调增加的。令 $\tau_n=\lim_{n\to\infty}\tau_n$，则有 $\tau_\infty\le\tau_e$ 几乎是处处成立的。我们需要证明 $\tau_\infty=\infty$，即 $\tau_e=\infty$ 且 $\left(x_1(t),x_2(t),y_1(t),y_2(t)\right)\in R_+^4$ 几乎处处成立。如若不然，则存在两个常数 $K_1>0$ 和 $e\in(0,1)$ 使得 $P(\tau_\infty\le K_1)\ge e$。于是存在 $n_1\in N$，且 $n_1>n_0$ 使得 $P(\tau_n\le K_1)\ge e$，$n\ge n_1$。

让

$$
\begin{aligned}
&V\left(x_1(t),x_2(t),y_1(t),y_2(t)\right)\\
&=\left(kx_1-1-k_1\ln x_1\right)+\left(y_1-1-\ln x_1\right)+\left(kx_2-1-k_2\ln x_2\right)+\left(y_2-1-\ln y_2\right).
\end{aligned}
$$

运用 Itô 公式，有

$$
\begin{aligned}
&\mathrm{d}V\left(x_1(t),x_2(t),y_1(t),y_2(t)\right)\\
&=LV\mathrm{d}t+\sigma_1\left(kx_1(t)-k_1\right)\mathrm{d}w(t)+\sigma_2\left(y_1(t)-1\right)\mathrm{d}w(t)+\sigma_3\left(kx_2(t)-k_2\right)\mathrm{d}w(t)+\sigma_4\left(y_2(t)-1\right)\mathrm{d}w(t)\\
&\quad+\int_z\left[k\psi_1(z)x_1(t)-k_1\ln\left(1+(\mathrm{d}t,\mathrm{d}z)\right)\right]+\int_z\left[\psi_2(z)y_1(t)-\ln\left(1+(\mathrm{d}t,\mathrm{d}z)\right)\right]\\
&\quad+\int_z\left[k\psi_3(z)x_2(t)-k_2\ln\left(1+(\mathrm{d}t,\mathrm{d}z)\right)\right]+\int_z\left[\psi_4(z)y_2(t)-\ln\left(1+(\mathrm{d}t,\mathrm{d}z)\right)\right]
\end{aligned}
\tag{2.2}
$$

其中

$$LV \le -\frac{kr_1}{K_1}x_1^2 + \left(kr_1 + \frac{k_1r_1}{K_1} + \frac{k_2r_2}{K_2}\right)x_1 + \left(\frac{ak_1}{b_{11}} + \frac{ak_2}{b_{21}} - c_1\right)y_1 + c_1 - k_1r_1$$
$$-\frac{kr_2}{K_2}x_2^2 + \left(kr_2 + \frac{k_1r_1}{K_1} + \frac{k_2r_2}{K_2}\right)x_2 + \left(\frac{ak_1}{b_{12}} + \frac{ak_2}{b_{22}} - c_2\right)y_2 + c_2 - k_2r_2$$
$$+\int_z\left[k_1\psi_1 - k_1\ln(1+\psi_1)\right]v(\mathrm{d}z) + \int_z\left[\psi_2 - \ln(1+\psi_2)\right]v(\mathrm{d}z) + \frac{k_1\sigma_1^2+\sigma_2^2}{2}$$
$$+\int_z\left[k_2\psi_3 - k_2\ln(1+\psi_3)\right]v(\mathrm{d}z) + \int_z\left[\psi_4 - \ln(1+\psi_4)\right]v(\mathrm{d}z) + \frac{k_2\sigma_3^2+\sigma_4^2}{2}.$$

选取适当 k,k_1,k_2 从而，获得

$$\begin{aligned}LV \le{}& \frac{K_1^2}{4k^2r_1^2}\left(kr_1 + \frac{k_1r_1}{K_1} + \frac{k_2r_2}{K_2}\right)^2 + c_1 - k_1r_1 + \frac{K_2^2}{4k^2r_2^2}\left(kr_2 + \frac{k_1r_1}{K_1} + \frac{k_2r_2}{K_2}\right) + c_2 - k_2r_2\\ &+\int_z\left[k_1\psi_1 - k_1\ln(1+\psi_1)\right]v(\mathrm{d}z) + \int_z\left[\psi_2 - \ln(1+\psi_2)\right]v(\mathrm{d}z) + \frac{k_1\sigma_1^2+\sigma_2^2}{2}\\ &+\int_z\left[k_2\psi_3 - k_2\ln(1+\psi_3)\right]v(\mathrm{d}z) + \int_z\left[\psi_4 - \ln(1+\psi_4)\right]v(\mathrm{d}z) + \frac{k_2\sigma_3^2+\sigma_4^2}{2}\end{aligned} \tag{2.3}$$

从而，可得 $LV \le C$，其中

$$C = \frac{K_1^2}{4k^2r_1^2}\left(kr_1 + \frac{k_1r_1}{K_1} + \frac{k_2r_2}{K_2}\right)^2 + c_1 - k_1r_1 + \frac{K_2^2}{4k^2r_2^2}\left(kr_2 + \frac{k_1r_1}{K_1} + \frac{k_2r_2}{K_2}\right) + c_2 - k_2r_2$$
$$+\int_z\left[k_1\psi_1 - k_1\ln(1+\psi_1)\right]v(\mathrm{d}z) + \int_z\left[\psi_2 - \ln(1+\psi_2)\right]v(\mathrm{d}z) + \frac{k_1\sigma_1^2+\sigma_2^2}{2}$$
$$+\int_z\left[k_2\psi_3 - k_2\ln(1+\psi_3)\right]v(\mathrm{d}z) + \int_z\left[\psi_4 - \ln(1+\psi_4)\right]v(\mathrm{d}z) + \frac{k_2\sigma_3^2+\sigma_4^2}{2}$$

对式子(2.2)两边从 0 到 $\tau_k \wedge T$ 之间积分，得到

$$\begin{aligned}0 &\le E\Big(V\big(x_1(\tau_k\wedge T), y_1(\tau_k\wedge T), x_2(\tau_k\wedge T), y_2(\tau_k\wedge T)\big)\Big)\\ &\le V\big(x_1(0), y_1(0), x_2(0), y_2(0)\big) + CT.\end{aligned} \tag{2.4}$$

对每一个 $g>0$，定义

$$G(g) = \inf\left\{V(x_1,y_1,x_2,y_2): x_i \ge g, y_i \ge g \vee x_i \le \frac{1}{g}, y_i \le \frac{1}{g}, i = 1,2\right\}$$

则有 $\lim_{g\to\infty} G(g) = \infty$。因此，令 $k\to\infty$，有 $\infty > V\big(x_1(0),y_1(0),x_2(0),y_2(0)\big) + CT = \infty$，这与之前的假设矛盾。因此，$\tau_\infty = \infty$，则具有 Lévy 驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)存在唯一的全局解 $\big(x_1(t),y_1(t),x_2(t),y_2(t)\big)$。定理证明完毕。

定理 2.2 假设对于任何初始条件 $\big(x_1(0),y_1(0),x_2(0),y_2(0)\big)$，如果下列条件成立

$$c_1 > 2ka + \int_z\left[\big(1+\psi_2(z)\big)^p - 1 - p\psi_2(z)\right]v(\mathrm{d}z),$$

$$c_2 > 2ka + \int_z\left[\big(1+\psi_4(z)\big)^p - 1 - p\psi_4(z)\right]$$

则具有 Lévy 跳跃驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)解是随机最终有界。

证明：定义

$$V\left(x_1(t),x_2(t),y_1(t),y_2(t)\right)=x_1^p+y_1^p+x_2^p+y_2^p,\left(x_1(t),x_2(t),y_1(t),y_2(t)\right)\in R_+^4.$$

运用 Itô 公式，得到

$$\begin{aligned}\mathrm{d}V\left(x_1,y_1,x_2,y_2\right)=&LV\mathrm{d}t+\sigma_1 px_1^p\mathrm{d}w+\sigma_2 py_1^p\mathrm{d}w+\sigma_3 px_2^p\mathrm{d}w+\sigma_4 py_2^p\mathrm{d}w\\&+\int_z\left[\left(1+\psi_1(z)\right)^p-1\right]x_1^p\tilde{N}(\mathrm{d}t,\mathrm{d}z)+\int_z\left[\left(1+\psi_2(z)\right)^p-1\right]y_1^p\tilde{N}(\mathrm{d}t,\mathrm{d}z)\\&+\int_z\left[\left(1+\psi_3(z)\right)^p-1\right]x_2^p\tilde{N}(\mathrm{d}t,\mathrm{d}z)+\int_z\left[\left(1+\psi_4(z)\right)^p-1\right]y_2^p\tilde{N}(\mathrm{d}t,\mathrm{d}z),\end{aligned}$$

其中

$$\begin{aligned}LV\le&-x_1^p+\left[-c_1+2ka+\int_z\left[\left(1+\psi_2(z)\right)^p-1-p\psi_2(z)\right]v(\mathrm{d}z)\right]y_1^p\\&-x_2^p+\left[-c_2+2ka+\int_z\left[\left(1+\psi_4(z)\right)^p-1-p\psi_4(z)\right]v(\mathrm{d}z)\right]y_2^p+2N\\&\le-\alpha\left(x_1^p+y_1^p+x_2^p+y_2^p\right)+2N=-\alpha V(t)+2N,\end{aligned}\tag{2.5}$$

且

$$-\frac{r_1}{K_1}x_1^{p+1}+\left(r_1+1+\int_z\left[\left(1+\psi_1(z)\right)^p-1-p\psi_1(z)\right]v(\mathrm{d}z)\right)x_1^p<N,$$

$$-\frac{r_2}{K_2}x_2^{p+1}+\left(r_2+1+\int_z\left[\left(1+\psi_3(z)\right)^p-1-p\psi_3(z)\right]v(\mathrm{d}z)\right)x_2^p<N,$$

$$\begin{aligned}\alpha=\min\Big\{&1,c_1-2ka-\int_z\left[\left(1+\psi_2(z)\right)^p-1-p\psi_2(z)\right]v(\mathrm{d}z),\\&c_2-2ka-\int_z\left[\left(1+\psi_4(z)\right)^p-1-p\psi_4(z)\right]v(\mathrm{d}z)\Big\}.\end{aligned}$$

运用 Itô 公式，得到

$$\begin{aligned}&\mathrm{d}\left[\mathrm{e}^{\alpha t}V\right]=\alpha\mathrm{e}^{\alpha t}V\mathrm{d}t+\mathrm{e}^{t}\mathrm{d}V\\&\le N\mathrm{e}^{\alpha t}\mathrm{d}t+\mathrm{e}^{\alpha t}\left(\sigma_1 px_1^p\mathrm{d}w+\sigma_2 py_1^p\mathrm{d}w+\sigma_3 px_2^p\mathrm{d}w+\sigma_4 py_2^p\mathrm{d}w\right)\\&\quad+\mathrm{e}^{\alpha t}\int_z\left[\left(1+\psi_1(z)\right)^p-1\right]x_1^p\tilde{N}(\mathrm{d}t,\mathrm{d}z)+\mathrm{e}^{\alpha t}\int_z\left[\left(1+\psi_2(z)\right)^p-1\right]y_1^p\tilde{N}(\mathrm{d}t,\mathrm{d}z)\\&\quad+\mathrm{e}^{\alpha t}\int_z\left[\left(1+\psi_3(z)\right)^p-1\right]x_2^p\tilde{N}(\mathrm{d}t,\mathrm{d}z)+\mathrm{e}^{\alpha t}\int_z\left[\left(1+\psi_4(z)\right)^p-1\right]y_2^p\tilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned}$$

对 $\mathrm{d}\left[\mathrm{e}^{\alpha t}V\right]$ 两边从 0 到 t 积分，然后取期望，则有

$$\mathrm{e}^{t}E\left[V(X)\right]\le V\left(x_1(0),x_2(0),y_1(0),y_2(0)\right)+N\mathrm{e}^{t}-N.\tag{2.6}$$

从而，则有

$$\lim_{t+\infty}\sup E\left[V(x)\right]\le N$$

定理证明完毕。

3. 灭绝性与持久性

本小节，主要讨论具有 Lévy 驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)的灭绝性和持久性。

定义：种群物种具有均值强持久性，如果 $\langle x(t)\rangle_*>0$，其中

$$\langle x(t)\rangle_*=\frac{1}{t}\int_0^t x(s)\mathrm{d}s,\ \langle x(t)\rangle_*=\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\mathrm{d}s$$

定理 3.1

(1) 如果满足下列条件之一

$$\langle \mathrm{I} \rangle \quad r_1 + \int_z \left[\ln\left(1+\psi_1(z)\right) - \psi_1(z) \right](\mathrm{d}z) > \frac{\sigma_1^2}{2}$$

$$\langle \mathrm{II} \rangle \quad r_2 + \int_z \left[\ln\left(1+\psi_3(z)\right) - \psi_3(z) \right](\mathrm{d}z) > \frac{\sigma_3^2}{2}.$$

则具有 Lévy 驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)有如下性质：

$$\left\langle x_1(t) \right\rangle_* > 0, \left\langle x_2(t) \right\rangle_* > 0,$$

$$\left\langle y_1(t) \right\rangle_* > 0, \left\langle y_2(t) \right\rangle_* > 0.$$

(2) 如果满足下列条件之一

$$\langle \mathrm{I} \rangle \quad c_1 + \frac{\sigma_2^2}{2} > \int_z \left[\ln\left(1+\psi_2(z)\right) - \psi_2(z) \right](\mathrm{d}z)$$

$$\langle \mathrm{II} \rangle \quad c_2 + \frac{\sigma_4^2}{2} > \int_z \left[\ln\left(1+\psi_4(z)\right) - \psi_4(z) \right](\mathrm{d}z).$$

则具有 Lévy 驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)有如下性质：

$$\left\langle x_1(t) \right\rangle_* > 0, \left\langle x_2(t) \right\rangle_* > 0.$$

(3) 此外，如果满足下列条件

$$c_1 + \frac{\sigma_2^2}{2} > 2ka + \int_z \left[\ln\left(1+\psi_2(z)\right) - \psi_2(z) \right](\mathrm{d}z).$$

则具有 Lévy 驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)有如下性质：

$$\lim_{t\to\infty} y_1(t) = 0.$$

和

如果满足下列条件

$$c_2 + \frac{\sigma_4^2}{2} > 2ka + \int_z \left[\ln\left(1+\psi_4(z)\right) - \psi_4(z) \right](\mathrm{d}z).$$

则具有 Lévy 驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)有如下性质：

$$\lim_{t\to\infty} y_2(t) = 0.$$

证明：定义

$$V(x_1) = \ln\left(x_1(t)\right). \tag{3.1}$$

运用 Itô 公式，得到

$$\begin{aligned} \mathrm{d}V(x_1) = & \left[r_1 - \frac{r_1}{K_1}(x_1 + x_2) - y_1 \frac{a}{b_{11} + x_1 + x_2} - y_2 \frac{a}{b_{12} + x_1 + x_2} \right] \mathrm{d}t \\ & - \left(\frac{\sigma_1^2}{2} - \int_z \ln\left(1+\psi_1(z)\right) - \psi_1(z) v(\mathrm{d}z) \right) \mathrm{d}t \\ & + \sigma_1 \mathrm{d}w(t) + \int_z \ln\left(1+\psi_1(z)\right) \tilde{N}(\mathrm{d}t, \mathrm{d}z). \end{aligned} \tag{3.2}$$

对式子(3.2)两边积分且除以 t，可获得

$$\frac{\ln\frac{x_1(t)}{x_1(0)}}{t} \ge r_1 - \frac{\sigma_1^2}{2} + \int_z \ln\left(1+\psi_1(z)\right) - \psi_1(z)v(\mathrm{d}z) - \frac{r_1}{K_1}\langle x_1 + x_2\rangle - \frac{a}{b_{11}}\langle y_1\rangle \\ - \frac{a}{b_{12}}\langle y_2\rangle + \sigma_1\frac{w(t)}{t} + \frac{1}{t}\int_0^t\int_z \ln\left(1+\psi_1(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z). \tag{3.3}$$

从而，可获得

$$\frac{r_1}{K_1}\langle x_1\rangle_* + \frac{r_1}{K_1}\langle x_2\rangle_* + \frac{a}{b_{11}}\langle y_1\rangle_* + \frac{a}{b_{12}}\langle y_2\rangle_* \ge r_1 - \frac{\sigma_1^2}{2} + \int_z\left[\ln\left(1+\psi_1(z)\right) - \psi_1(z)\right](\mathrm{d}z).$$

定义

$$V(y_1) = \ln\left(y_1(t)\right)$$

运用 Itô 公式，可得到

$$\mathrm{d}V(y_1) = \left[-c_1 + kx_1\frac{a}{b_{11}+x_1+x_2} + kx_2\frac{a}{b_{21}+x_1+x_2}\right]\mathrm{d}t \\ -\left(\frac{\sigma_2^2}{2} - \int_z \ln\left(1+\psi_2(z)\right) - \psi_2(z)v(\mathrm{d}z)\right)\mathrm{d}t \\ + \sigma_2\mathrm{d}w(t) + \int_z \ln\left(1+\psi_2(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z). \tag{3.5}$$

式子(3.4)两边积分且除以 t，可获得

$$\frac{\ln\frac{y_1(t)}{y_1(0)}}{t} \le \frac{ka}{b_{11}}\langle x_1\rangle + \frac{ka}{b_{21}}\langle x_2\rangle - c_1 - \frac{\sigma_2^2}{2} + \int_z \ln\left(1+\psi_2(z)\right) - \psi_2(z)v(\mathrm{d}z) \\ + \sigma_2\frac{w(t)}{t} + \frac{1}{t}\int_0^t\int_z \ln\left(1+\psi_2(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z). \tag{3.6}$$

从而，我们有

$$c_1 + \frac{\sigma_2^2}{2} - \int_z \ln\left(1+\psi_2(z)\right) - \psi_2(z)v(\mathrm{d}z) \le \frac{ka}{b_{11}}\langle x_1\rangle_* + \frac{ka}{b_{21}}\langle x_2\rangle_*$$

定义

$$V(x_2) = \ln\left(x_2(t)\right)$$

运用 Itô 公式，可得到：

$$\mathrm{d}V(x_2) = \left[r_2 - \frac{r_2}{K_2}(x_1+x_2) - y_1\frac{a}{b_{21}+x_1+x_2} - y_2\frac{a}{b_{22}+x_1+x_2}\right]\mathrm{d}t \\ -\left(\frac{\sigma_3^2}{2} - \int_z \ln\left(1+\psi_3(z)\right) - \psi_3(z)v(\mathrm{d}z)\right)\mathrm{d}t \\ + \sigma_3\mathrm{d}w(t) + \int_z \ln\left(1+\psi_3(z)\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z). \tag{3.8}$$

对式子(3.8)两边积分且除以 t，可获得

$$\frac{\ln \frac{x_2(t)}{x_2(0)}}{t} \geq r_2 - \frac{r_2}{K_2}\langle x_1 + x_2 \rangle - \frac{a}{b_{21}}\langle y_1 \rangle - \frac{a}{b_{22}}\langle y_2 \rangle$$
$$- \frac{\sigma_3^2}{2} + \int_z \ln(1+\psi_3(z)) - \psi_3(z)\nu(\mathrm{d}z) \tag{3.9}$$
$$+ \sigma_3 \frac{w(t)}{t} + \frac{1}{t}\int_0^t \int_z \ln(1+\psi_3(z))\tilde{N}(\mathrm{d}t, \mathrm{d}z).$$

从而，可获得

$$\frac{r_2}{K_2}\langle x_1 \rangle_* + \frac{r_2}{K_2}\langle x_2 \rangle_* + \frac{a}{b_{21}}\langle y_1 \rangle_* + \frac{a}{b_{22}}\langle y_2 \rangle_* \geq r_2 - \frac{\sigma_3^2}{2} + \int_z [\ln(1+\psi_3(z)) - \psi_3(z)](\mathrm{d}z).$$

定义

$$V(y_2) = \ln(y_2(t)) \tag{3.10}$$

运用 Itô 公式，可得到：

$$\mathrm{d}V(y_2) = \left[-c_2 + kx_1 \frac{a}{b_{12} + x_1 + x_2} + kx_2 \frac{a}{b_{22} + x_1 + x_2}\right]\mathrm{d}t$$
$$-\left(\frac{\sigma_4^2}{2} - \int_z \ln(1+\psi_4(z)) - \psi_4(z)\nu(\mathrm{d}z)\right)\mathrm{d}t \tag{3.11}$$
$$+ \sigma_4 \mathrm{d}w(t) + \int_z \ln(1+\psi_4(z))\tilde{N}(\mathrm{d}t, \mathrm{d}z).$$

对式子(3.11)两边积分且除以 t，可获得

$$\frac{\ln \frac{y_2(t)}{y_2(0)}}{t} \geq -c_2 + \frac{ka}{b_{12}}\langle x_1 \rangle + \frac{ka}{b_{22}}\langle x_2 \rangle - \frac{\sigma_4^2}{2} + \int_z [\ln(1+\psi_4(z)) - \psi_4(z)](\mathrm{d}z) \tag{3.12}$$
$$+ \sigma_4 \frac{w(t)}{t} + \frac{1}{t}\int_0^t \int_z \ln(1+\psi_4(z))\tilde{N}(\mathrm{d}t, \mathrm{d}z).$$

从而，我们有

$$c_2 + \frac{\sigma_4^2}{2} - \int_z [\ln(1+\psi_4(z)) - \psi_4(z)]\nu(\mathrm{d}z) \leq \frac{ka}{b_{12}}\langle x_1 \rangle_* + \frac{ka}{b_{22}}\langle x_2 \rangle_*.$$

定理证明完毕。

4. 平稳分布

本小节，主要讨论具有 Lévy 跳跃驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)存在唯一的平稳分布。

引理 4.1 设 $X_1(t) = X_1(x_1, y_1, x_2, y_2, t)$ 和 $X_2(t) = X_2(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, t)$ 是模型(1.3)的任意两个解，如果满足下列条件

$$\frac{1}{K_1} > \frac{2ka}{b_{11}} + \frac{ka}{b_{21}} + \frac{2ka}{b_{12}} + \frac{ka}{b_{22}}, \quad \frac{1}{K_2} > \frac{ka}{b_{11}} + \frac{2ka}{b_{21}} + \frac{ka}{b_{12}} + \frac{2ka}{b_{22}}$$

和

$$k > \max\left\{\frac{1}{b_{12}} - \frac{1}{b_{22}}, \frac{1}{b_{11}} - \frac{1}{b_{21}}\right\}$$

则有

$$\lim_{t\to+\infty}\left(\left\|X_1(t)-X_2(t)\right\|\right)=0$$

即

$$\lim_{t\to+\infty}\left(E\left\|x_1(t)-\tilde{x}_1(t)\right\|+E\left\|y_1(t)-\tilde{y}_1(t)\right\|+E\left\|x_2(t)-\tilde{x}_2(t)\right\|+E\left\|y_2(t)-\tilde{y}_2(t)\right\|\right)=0$$

证明：定义

$$V(x_1,y_1,x_2,y_2)=\alpha\left|\ln x_1-\ln\tilde{x}_1\right|+\left|\ln y_1-\ln\tilde{y}_1\right|+\beta\left|\ln x_2-\ln\tilde{x}_2\right|+\left|\ln y_2-\ln\tilde{y}_2\right|,$$

则，我们有

$$\begin{aligned}
&\mathrm{d}V(x_1,y_1,x_2,y_2,\tilde{x}_1,\tilde{y}_1,\tilde{x}_2,\tilde{y}_2)\\
&=\mathrm{d}\left[\alpha\left|\ln x_1-\ln\tilde{x}_1\right|+\left|\ln y_1-\ln\tilde{y}_1\right|+\beta\left|\ln x_2-\ln\tilde{x}_2\right|+\left|\ln y_2-\ln\tilde{y}_2\right|\right]\\
&\le\left\{\left(-\frac{1}{K_1}+\frac{1}{K_2}+\frac{2ka}{b_{11}}+\frac{ka}{b_{21}}+\frac{2ka}{b_{12}}+\frac{ka}{b_{22}}\right)\left|x_1-\tilde{x}_1\right|\right.\\
&\quad+\left(\frac{1}{K_1}-\frac{1}{K_2}+\frac{ka}{b_{11}}+\frac{2ka}{b_{21}}+\frac{ka}{b_{12}}+\frac{2ka}{b_{22}}\right)\left|x_2-\tilde{x}_2\right|\\
&\quad+\left(\frac{a}{b_{11}}+\frac{a}{b_{21}}-ka\right)\left|y_1-\tilde{y}_1\right|+\left(\frac{a}{b_{11}}+\frac{a}{b_{21}}+ka\right)(y_1+\tilde{y}_1)\\
&\quad\left.+\left(\frac{a}{b_{12}}+\frac{a}{b_{22}}-ka\right)\left|y_2-\tilde{y}_2\right|+\left(\frac{a}{b_{12}}+\frac{a}{b_{22}}+ka\right)(y_2+\tilde{y}_2)\right\}\mathrm{d}t.
\end{aligned}\tag{4.1}$$

因此，我们有

$$\begin{aligned}
EV(x_1,y_1,x_2,y_2)\le&\left(-\frac{1}{K_1}+\frac{1}{K_2}+\frac{2ka}{b_{11}}+\frac{ka}{b_{21}}+\frac{2ka}{b_{12}}+\frac{ka}{b_{22}}\right)\int_0^t E\left|x_1-\tilde{x}_1\right|\mathrm{d}t\\
&+\left(\frac{1}{K_1}-\frac{1}{K_2}+\frac{ka}{b_{11}}+\frac{2ka}{b_{21}}+\frac{ka}{b_{12}}+\frac{2ka}{b_{22}}\right)\int_0^t E\left|x_2-\tilde{x}_2\right|\mathrm{d}t\\
&+\left(\frac{a}{b_{11}}+\frac{a}{b_{21}}-ka\right)\int_0^t E\left|y_1-\tilde{y}_1\right|\mathrm{d}t+\left(\frac{a}{b_{11}}+\frac{a}{b_{21}}+ka\right)Nt\\
&+\left(\frac{a}{b_{12}}+\frac{a}{b_{22}}-ka\right)\int_0^t E\left|y_2-\tilde{y}_2\right|\mathrm{d}t+\left(\frac{a}{b_{12}}+\frac{a}{b_{22}}+ka\right)Nt.
\end{aligned}\tag{4.2}$$

由于 $EV(x_1,y_1,x_2,y_2)>0$，因此，我们获得

$$\begin{aligned}
&\left(\frac{1}{K_1}-\frac{2ka}{b_{11}}-\frac{ka}{b_{21}}-\frac{2ka}{b_{12}}-\frac{ka}{b_{22}}\right)\int_0^t E\left|x_1-\tilde{x}_1\right|\mathrm{d}t\\
&+\left(\frac{1}{K_2}-\frac{ka}{b_{11}}-\frac{2ka}{b_{21}}-\frac{ka}{b_{12}}-\frac{2ka}{b_{22}}\right)\int_0^t E\left|x_2-\tilde{x}_2\right|\mathrm{d}t\\
&+\left(ka-\frac{a}{b_{11}}-\frac{a}{b_{21}}\right)\int_0^t E\left|y_1-\tilde{y}_1\right|\mathrm{d}t+\left(ka-\frac{a}{b_{12}}-\frac{a}{b_{22}}\right)\int_0^t E\left|y_2-\tilde{y}_2\right|\mathrm{d}t\\
&\le V(0)+\left(\frac{a}{b_{11}}+\frac{a}{b_{21}}+ka\right)Nt+\left(\frac{a}{b_{12}}+\frac{a}{b_{22}}+ka+\frac{2}{K_1}+\frac{2}{K_2}\right)Nt<\infty.
\end{aligned}\tag{4.3}$$

因此

$$E|x_1-\tilde{x}_1|, E|y_1-\tilde{y}_1|, E|x_2-\tilde{x}_2|, E|y_2-\tilde{y}_2| \in L[0,+\infty).$$

类似文献[23]定理 6.2 的证明，我们可得

$$\lim_{t\to+\infty}\left(E\|x_1(t)-\tilde{x}_1(t)\|+E\|y_1(t)-\tilde{y}_1(t)\|+E\|x_2(t)-\tilde{x}_2(t)\|+E\|y_2(t)-\tilde{y}_2(t)\|\right)=0$$

几乎必然成立。引理证明完毕。

引理 4.2 对于任意 $p>0$，任意的紧子集 $D\subseteq\Re_+^4$，则有

$$\sup_{D\subseteq\Re_+^4} E\left[\sup_{0\le s\le t}|x_1(t)|^p\right]<+\infty, \sup_{D\subseteq\Re_+^4} E\left[\sup_{0\le s\le t}|y_1(t)|^p\right]<+\infty,$$

$$\sup_{D\subseteq\Re_+^4} E\left[\sup_{0\le s\le t}|x_2(t)|^p\right]<+\infty, \sup_{D\subseteq\Re_+^4} E\left[\sup_{0\le s\le t}|y_2(t)|^p\right]<+\infty.$$

证明：根据系统(1.3)，我们有

$$\begin{aligned}
x_1(t)=x_1(0)+\int_0^t x_1(s)\left[r_1\left(1-\frac{x_1(s)+x_2}{K_1(s)}\right)-y_1(s)\frac{a}{b_{11}+x_1(s)+x_2(s)}-y_2(s)\frac{a}{b_{12}+x_1(s)+x_2(s)}\right]\mathrm{d}s\\
+\int_0^t\sigma_1x_1(s)\mathrm{d}w(s)+\int_0^t\int_z\psi_1(z)x_1(s)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\\
y_1(t)=y_1(0)+\int_0^t y_1(s)\left[-c_1+kx_1(s)\frac{a}{b_{11}+x_1(s)+x_2(s)}+kx_2(s)\frac{a}{b_{21}+x_1(s)+x_2(s)}\right]\mathrm{d}s\\
+\int_0^t\sigma_2y_1(s)\mathrm{d}w(t)+\int_0^t\int_z\psi_2(z)y_1(s)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\\
x_2(t)=x_2(0)+\int_0^t x_2(s)\left[r_2\left(1-\frac{x_1(s)+x_2(s)}{K_2}\right)y_1(s)\frac{a}{b_{21}+x_1(s)+x_2(s)}-y_2(s)\frac{a}{b_{22}+x_1(s)+x_2(s)}\right]\mathrm{d}s\\
+\int_0^t\sigma_3x_2(s)\mathrm{d}w(t)+\int_0^t\int_z\psi_3(z)x_2(s)\tilde{N}(\mathrm{d}t,\mathrm{d}z),\\
y_2(t)=y_2(0)+\int_0^t y_2(s)\left[-c_2+kx_1(s)\frac{a}{b_{12}+x_1(s)+x_2(s)}+kx_2(s)\frac{a}{b_{22}+x_1(s)+x_2(s)}\right]\mathrm{d}s\\
+\int_0^t\sigma_4y_2(s)\mathrm{d}w(t)+\int_0^t\int_z\psi_4(z)y_2(s)\tilde{N}(\mathrm{d}t,\mathrm{d}z).
\end{aligned}\tag{4.4}$$

结合 Hölder 不等式，对于 $k=1,2,\cdots$，我们获得

$$\begin{aligned}
&E\left[\sup_{(k-1)\xi\le s\le k\xi}|x_1(s)|^p\right]\\
&\le 4^{p-1}|x_1((k-1)\xi)|^p+4^{p-1}E\left[\sup_{(k-1)\xi\le s\le k\xi}\left|\int_{(k-1)\xi}^{k\xi}x_1(s)\left[r_1\left(1-\frac{x_1(s)+x_2(s)}{K_1}\right)\right.\right.\right.\\
&\left.\left.\left.-y_1(s)\frac{a}{b_{11}+x_1(s)+x_2(s)}-y_2(s)\frac{a}{b_{12}+x_1(s)+x_2(s)}\right]\mathrm{d}s\right|^p\right]\\
&+4^{p-1}E\left[\sup_{(k-1)\xi\le s\le k\xi}\left|\int_{(k-1)\xi}^{k\xi}\sigma_1x_1(s)\mathrm{d}w(s)\right|^p\right]\\
&+4^{p-1}E\left[\sup_{(k-1)\xi\le s\le k\xi}\left|\int_{(k-1)\xi}^{k\xi}\int_z\psi_1(z)x_1(s)\tilde{N}(\mathrm{d}t,\mathrm{d}z)\right|^p\right].
\end{aligned}\tag{4.5}$$

根据定理 2.2，存在一个正数 $K(p)$，使得 $E\left|x_1(t)\right|^p, E\left|y_1(t)\right|^p, E\left|x_2(t)\right|^p, E\left|y_2(t)\right|^p \le K(p)$，从而，我们获得

$$
\begin{aligned}
&E\left[\sup_{(k-1)\xi\le s\le k\xi}\left|\int_{(k-1)\xi}^{k\xi} x_1(s)\left[r_1\left(1-\frac{x_1(s)+x_2(s)}{K_1}\right)\right.\right.\right.\\
&\left.\left.\left.-y_1(s)\frac{a}{b_{11}+x_1(s)+x_2(s)}-y_2(s)\frac{a}{b_{12}+x_1(s)+x_2(s)}\right]\mathrm{d}s\right|^p\right]\\
&\le\frac{1}{2}\xi^p E\left|x_1(s)\right|^{2p}+\frac{1}{2}\xi^p 5^{2p-1}\left[r_1^{2p}+\frac{r_1^{2p}}{K_1^{2p}}E\sup_{(k-1)\xi\le s\le k\xi}\left|x_1(s)\right|^{2p}+\frac{r_1^{2p}}{K_1^{2p}}E\sup_{(k-1)\xi\le s\le k\xi}\left|x_2(s)\right|^{2p}\right.\\
&\left.+\frac{a}{b_{11}}E\sup_{(k-1)\xi\le s\le k\xi}\left|y_1(s)\right|^{2p}+\frac{a}{b_{12}}\sup_{(k-1)\xi\le s\le k\xi}\left|y_2(s)\right|^{2p}\right]\\
&:=M_1(p)\xi^p.
\end{aligned}
\tag{4.6}
$$

根据 Burkholder-Davis-Gundy 不等式，我们有

$$
E\left[\sup_{(k-1)\xi\le s\le k\xi}\left|\int_{(k-1)\xi}^{k\xi}\sigma_1 x_1(s)\mathrm{d}w(s)\right|^p\right]\le C_p\xi^{\frac{p}{2}}\sigma_1^p E\left(\sup_{(k-1)\xi\le s\le k\xi}\left|x_1(s)\right|^p\right):=M_2(p)\xi^{\frac{p}{2}}. \tag{4.7}
$$

根据 Kunita’s 第一不等式，我们有

$$
\begin{aligned}
&E\left[\sup_{(k-1)\xi\le s\le k\xi}\left|\int_{(k-1)\xi}^{k\xi}\int_z \psi_1(z)x_1(s)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|^p\right]\\
&\le D_p\left\{E\left(\int_{(k-1)\xi}^{k\xi}\int_z\left|\psi_1(z)x_1(s)\right|^2 v(\mathrm{d}z)\mathrm{d}s\right)^{\frac{p}{2}}+E\left(\int_{(k-1)\xi}^{k\xi}\int_z\left|\psi_1(z)x_1(s)\right|^p v(\mathrm{d}z)\mathrm{d}s\right)\right\}\\
&\le D_p\xi^{\frac{p}{2}}G_1^{\frac{p}{2}}K(p)+D_p\xi G_2^{\frac{p}{2}}K(p).
\end{aligned}
\tag{4.8}
$$

根据(4.5)~(4.8)，我们有

$$
\sup_{D\subseteq\mathfrak{R}_+^4}E\left[\sup_{0\le s\le t}\left|x_1(t)\right|^p\right]<+\infty,\ \forall s\in[0,t],\ \forall t\ge 0.
$$

类似的证明，我们可以获得

$$
\sup_{D\subseteq\mathfrak{R}_+^4}E\left[\sup_{0\le s\le t}\left|y_1(t)\right|^p\right]<+\infty,\ \sup_{D\subseteq\mathfrak{R}_+^4}E\left[\sup_{0\le s\le t}\left|x_2(t)\right|^p\right]<+\infty,\ \sup_{D\subseteq\mathfrak{R}_+^4}E\left[\sup_{0\le s\le t}\left|y_2(t)\right|^p\right]<+\infty..
$$

引理证明完毕。

让 $C_b\left(\mathfrak{R}_+^4\right)$ 表示在 $\mathfrak{R}_+^4$ 上所有有界连续实值函数的集合，$P\left(\mathfrak{R}_+^4\right)$ 是在 $\left(\mathfrak{R}_+^4,\mathcal{B}\left(\mathfrak{R}_+^4\right)\right)$ 上所有概率测度的空间，其中 $\mathcal{B}\left(\mathfrak{R}_+^4\right)$ 是在 $\mathfrak{R}_+^4$ 上的 Borel σ-代数。对于系统(1.3)的任意解 $X\left(x_1,y_1,x_2,y_2\right)$，众所周知，$X\left(x_1(t),y_1(t),x_2(t),y_2(t)\right)$ 在 $\mathfrak{R}_+^4$ 上是一个马尔可夫过程。

定义 4.1 [24]：系统(1.3)的解 $X=(t,\phi)X\left(x_1,y_1,x_2,y_2\right)$ 是平稳分布，如果存在概率测度 $\mu\in P\left(\mathfrak{R}_+^4\right)$ 满足

$$
\mu(f)=\mu(\mathbb{P}_t f),t\ge 0
$$

其中

$$
\mu(f):=\int_{\mathfrak{R}_+^4} f(\phi)\mu(\mathrm{d}\phi)\text{ 和 }\mathbb{P}_t f(\phi)=Ef\left(X(t,\phi)\right),\ f\in C_b\left(\mathfrak{R}_+^4\right)
$$

对于 $\mu_1,\mu_2\in P(\Re_+^4)$，在 $P(\Re_+^4)$ 上定义度量空间

$$d(\mu_1,\mu_2)=\sup_{f\in\mathcal{M}}\left|\int_{\Re_+^4} f(\phi)\mu_1(\mathrm{d}\phi)-\int_{\Re_+^4} f(\psi)\mu_2(\mathrm{d}\psi)\right|$$

其中

$$\mathcal{M}:=\left\{f:\Re_+^4\to\Re,\left|f(\phi)-f(\psi)\right|\le\|\phi-\psi\|_{\Re_+^4},\forall\varphi,\psi\in\Re_+^4,\left|f(\cdot)\right|\le 1\right\}$$

则众所周知，在度量 $d(\cdot,\cdot)$ 上 $P(\Re_+^4)$ 是完备的。

定理 4.3 如果满足下列条件

$$\frac{1}{K_1}>\frac{2ka}{b_{11}}+\frac{ka}{b_{21}}+\frac{2ka}{b_{12}}+\frac{ka}{b_{22}},\ \frac{1}{K_2}>\frac{ka}{b_{11}}+\frac{2ka}{b_{21}}+\frac{ka}{b_{12}}+\frac{2ka}{b_{22}}$$

和

$$k>\max\left\{\frac{1}{b_{12}}-\frac{1}{b_{22}},\frac{1}{b_{11}}-\frac{1}{b_{21}}\right\}.$$

则具有 Lévy 跳跃驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)存在唯一的平稳分布。

证明：对任意给定 $\phi\in\Re_+^4$ 和 $\varepsilon>0$，下面证明存在一个时间 $T>0$ 使得

$$d(\mathbb{P}(\phi,t+s,\cdot),\mathbb{P}(\phi,t,\cdot))=\sup_{f\in\mathcal{M}}\left|Ef(X(t+s,\phi))-Ef(X(t,\phi))\right|\le\varepsilon,\tag{4.9}$$

对任意 $t\ge T$ 和 $s>0$。

事实上，对于任意 $f\in\mathcal{M}$ 和 $t,s>0$，我们获得

$$\begin{aligned}&\left|E\left[Ef(X(t+s,\phi))\,|\,\mathcal{F}_s\right]-Ef(X(t,\phi))\right|\\&=\left|Ef(X(t+s,\phi))-Ef(X(t,\phi))\right|\\&=\left|\int_{\Re_+^4}E(X(t,\psi))\mathbb{P}(\phi,s,\mathrm{d}\psi)-Ef(X(t,\phi))\right|\\&\le\int_{\Re_+^4}\left|Ef(X(t,\psi))-Ef(X(t,\phi))\right|\mathbb{P}(\phi,s,\mathrm{d}\psi)\\&\le 2\mathbb{P}(\phi,s,\Re_+^{4,c})+\int_{\Re_{+,R}^4}\left|Ef(X(t,\psi))-Ef(X(t,\phi))\right|\mathbb{P}(\phi,s,\mathrm{d}\psi)\end{aligned}\tag{4.10}$$

其中 $\Re_{+,R}^4=\left\{\phi\in\Re_+^4:\|\phi\|\le R\right\}$ 和 $\Re_{+,R}^{4,C}=\Re_+^4-\Re_{+,R}^4$，根据引理 3.2 以及引理 5.1 (参考文献[24])，存在一个充分大的 R 使得

$$\mathbb{P}(\phi,s,\Re_+^{4,c})\le\frac{\varepsilon}{4},\ \forall s>0.\tag{4.11}$$

另一方面，根据引理 3.1 以及引理 5.1 (参考文献[24])，存在一个时间 T_2 使得

$$\sup_{f\in\mathcal{M}}\left|Ef(X(t,\psi))-Ef(X(t,\phi))\right|\le\frac{\varepsilon}{4},\ \phi,\psi\in\Re_+^4,\ \forall t>T_2\tag{4.12}$$

因此，把(4.11)和(4.12)代入(4.10)，得到

$$\left|E\left[Ef(X(t+s,\phi))\,|\,\mathcal{F}_s\right]-Ef(X(t,\phi))\right|\le\varepsilon,\ \forall t>T_2,\ s>0\tag{4.13}$$

由于 $f\in\mathcal{M}$ 的任意性，我们获得式子(4.9)成立，即

$$d(\mathbb{P}(\phi,t+s,\cdot),\mathbb{P}(\phi,t,\cdot))\le\varepsilon.$$

因此，系统(1.3)的解 $X=(t,\phi)X(x_1,y_1,x_2,y_2)$ 的转移概率 $\mathbb{P}(\phi,t,\cdot)$ 弱收敛到某些测度 $\mu\in P(\mathfrak{R}_+^4)$。另一方面，对于任意 $f\in C_b(\mathfrak{R}_+^4)$， $X(x_1(t),y_1(t),x_2(t),y_2(t))$ 具有马尔可夫性质使得

$$\mathbb{P}_{t+s}f(\phi)=\mathbb{P}_t\mathbb{P}_s f(\phi),t,s\geq 0,\phi\in\mathfrak{R}_+^4.$$

因此，对于给定 $t\geq 0$，当 $s\to\infty$，则有

$$\mu(f)=\mu(\mathbb{P}_t f),\ f\in C_b(\mathfrak{R}_+^4)$$

故 μ 是系统(1.3)解 $X=(t,\phi)X(x_1,y_1,x_2,y_2)$ 的一个平稳分布。

下面我们将证明平稳分布的唯一性，如果 μ' 是系统(1.3)解 $X=(t,\phi)X(x_1,y_1,x_2,y_2)$ 的另一个平稳分布。让 $f\in C_{LB}(\mathfrak{R}_+^4)$，其中 $C_{LB}(\mathfrak{R}_+^4)$ 表示在 $\mathfrak{R}_+^4$ 上所有有界和 Lipschitz 连续函数族。根据引理 3.1，引理 3.2 和 hölder 不等式以及 $\mu,\mu'\in P(\mathfrak{R}_+^4)$ 的不变性，我们有

$$\left|\mu(f)-\mu'(f)\right|\leq\int_{\mathfrak{R}_+^4}\left|\mathbb{P}_t f(\phi)-\mathbb{P}_t f(\psi)\right|\mathrm{d}\mu(\phi)\mathrm{d}\mu'(\psi)<\varepsilon$$

这意味着存在唯一性的平稳分布，即具有 Lévy 跳跃驱动的 Rosenzweig-MacArthur 捕食者－食饵模型(1.3)存在唯一的平稳分布。定理证明完毕。

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