Pure Mathematics 理论数学, 2024, 14(5), 377-393
Published Online May 2024 in Hans. https://www.hanspub.org/journal/pm
https://doi.org/10.12677/pm.2024.145194

基于几何QSR-耗散性和线性静态输出反馈控制器的非线性随机离散系统的几何镇定

杨 帆，温鑫湲，任院红*，白雪洁

河北农业大学理学院，河北 保定

收稿日期：2024年4月10日；录用日期：2024年5月11日；发布日期：2024年5月31日

摘 要

本文研究了非线性随机离散系统的几何镇定问题。首先，引入了非线性随机离散系统的几何随机增量QSR耗散的概念，并给出了该概念的线性矩阵不等式(LMI)表示形式；其次，基于几何随机增量QSR耗散性和线性静态输出反馈控制器，提出了该系统的概率意义下的几何均方增量稳定的充要条件。最后，通过进行数值模拟，展示了所得结论的有效性。

关键词

几何随机增量QSR耗散，几何均方增量稳定，线性静态输出反馈控制器

Geometric Stabilization of Nonlinear Stochastic Discrete-Time Systems Based on Geometric QSR-Dissipativity and Linear Static Output Feedback Controller

Fan Yang, Xinyuan Wen, Yuanhong Ren*, Xuejie Bai

College of Science, Heibei Agricultural University, Baoding Hebei

Received: Apr. 10^{th}, 2024; accepted: May 11^{th}, 2024; published: May 31^{st}, 2024

* 通讯作者。

文章引用: 杨帆, 温鑫湲, 任院红, 白雪洁. 基于几何QSR-耗散性和线性静态输出反馈控制器的非线性随机离散系统的几何镇定[J]. 理论数学, 2024, 14(5): 377-393. DOI: 10.12677/pm.2024.145194

Abstract

In this paper, the geometric stabilization problem of nonlinear stochastic discrete-time systems is studied. Firstly, the concept of geometric stochastic incremental QSR dissipativity for nonlinear stochastic discrete-time systems is introduced, and the expression of the concept of linear matrix inequality (LMI) is given. Secondly, based on the geometric stochastic incremental QSR dissipativity and the linear static output feedback controller, the sufficient and necessary conditions for the geometric mean square incremental stability in probability of the system are proposed. Finally, the validity of the results is demonstrated by numerical simulation.

Keywords

Geometrically Stochastically Incrementally QSR-Dissipative, Geometrically Mean Square Stable, Linear Static Output Feedback Controller

1. 引言

在控制工程中, 耗散理论为动力系统的分析和控制设计提供了一个基本框架 [1], 该框架基于系统能量相关考虑, 使用输入, 状态和输出系统描述来设计动力系统. 动力系统的耗散假设对系统的动力学行为有一个基本的约束, 耗散动力系统的存储能量最多等于系统中存储的初始能量与系统外部提供的总能量的总和. Willems [2] [3]在关于耗散动力系统的两部分论文中开创性地给出了发展一般连续时间非线性确定性动力系统耗散理论的关键基础. 耗散理论在解决鲁棒性, 风险敏感干扰抑制, 反馈互连稳定性和最优性方面发挥了重要作用 [4]. [5]为非线性随机动力系统的分析和控制设计提供了一般框架. 在 [6]中, 作者提出了一类非线性随机离散系统的有限增益非扩张性理论, 将耗散理论扩展到了非线性随机离散系统, 对于该系统的增量耗散性的研究并不多见.

稳定性是众多学者研究动力系统的核心内容, 对于增量稳定性的研究是其中的一个重要方面. 在研究非线性系统的增量稳定性时, 要求动力系统的所有轨迹彼此收敛 [7]. 近年来, 增量稳定性在非线性系统观测器设计 [8], 跟踪设计 [9]等领域有广泛的应用前景. 据作者所知, 一个涉及非线性随

机离散系统概率意义下几何均方增量稳定性与几何随机增量QSR耗散性的关系理论, 并没有相关文献提及.

本文对非线性随机离散系统展开研究, 首先介绍该系统概率意义下几何均方增量稳定性 [10]和几何随机增量QSR耗散的概念, 给出几何随机增量QSR 耗散的LMI形式; 随后, 通过设计一个线性静态输出反馈控制器, 在适当的存能函数和供给率下, 提出基于几何随机增量QSR耗散和线性静态输出反馈控制器的随机离散系统几何均方增量稳定的充分必要条件.

2. 预备知识

$\mathbb{R}$是实数集, $\overline{\mathbb{R}}_+$是非负实数集, $\mathbb{R}_-$和$\mathbb{R}_+$分别表示负实数集和正实数集, $\mathbb{R}^n$为实列向量, $\mathbb{R}^{n\times m}$是$n\times m$维实矩阵. $\mathbb{N}^m$是$m\times m$维的半正定矩阵. $||\cdot||$是向量的欧几里得范数. $V_s\in L^1$ 代表连续可微函数: $\mathbb{R}^n\to\mathbb{R}$. $\mathbb{Z}^+$是正整数集. $\overline{\mathbb{Z}}_+$是非负整数集. $\mathbb{S}^m$代表$m\times m$ 维的对称矩阵. $(\Omega,\mathcal{F},\mathcal{P})$是全概率空间, 其中$\Omega$表示样本空间, $\mathcal{F}$表示Ω的子集的σ-代数, $\mathcal{P}$定义为σ-代数$\mathcal{F}$上的概率测度.

考虑以下的非线性随机动态系统

$$\begin{cases} x(k+1)=f(x(k))+G(x(k))u(k)+D(x(k))\omega(k), x(0)=x_0, \\ y(k)=h(x(k)), \end{cases} \tag{1}$$

其中, 对于所有的$k\in\overline{\mathbb{Z}}_+$, $x(k)\in\mathcal{D}\subseteq\mathbb{R}^n$, $\mathcal{D}$是一个开集, 且$0\in\mathcal{D}$, $u(k)\in\mathbb{U}\subseteq\mathbb{R}^m$, $y(k)\in Y\subseteq\mathbb{R}^l$ 分别为$\mathcal{D}$-值, $\mathcal{U}$-值, Y-值的随机过程. $w(k), k\in\overline{\mathbb{Z}}_+$为$(\Omega,\mathcal{F},\mathcal{P})$中的d 维独立同分布随机过程, 并且$w(k)$是零均值的. f: $\mathcal{D}\to\mathbb{R}^n$, G: $\mathcal{D}\to\mathbb{R}^{n\times m}$, D: $\mathcal{D}\to\mathbb{R}^{n\times d}$ 和h: $\mathcal{D}\to\mathbb{R}^l$. 对于定义在状态空间$\mathcal{D}$, $\mathcal{U}$和$\mathcal{Y}$上的系统(1), 定义一个输入和输出状态空间, 分别由带有指数集$\overline{\mathbb{Z}}_+$ 的$\mathcal{U}$-值和Y- 值随机过程组成. 假设$f(\cdot)$, $G(\cdot)$, $D(\cdot)$ 和$h(\cdot)$是连续可微函数, 并且$f(\cdot)$ 至少有一个平衡点, 使得$f(0)=0$, $G(0)=0$, $D(0)=0$ 和$h(0)=0$. 考虑系统(1)并且令$V_s:\mathcal{D}\to\mathbb{R}$. 然后, 定义$x$ 和$\hat{x}$ 的差分算子为

$$\Delta V_s(x(k),\hat{x}(k))=E[V_s(x(k+1),\hat{x}(k+1))]-V_s(x(k),\hat{x}(k)), \tag{2}$$

这里, 定义中的差分算子是一个确定性函数, 它不涉及系统状态的期望, 而只涉及$\omega=\omega(k)$的期望, 其中$k\in\overline{\mathbb{Z}}_+$, $x=x(k)$, $\hat{x}=\hat{x}(k)$. 当差分算子沿状态向量$x(k)$, $\hat{x}(k)$ 的分布求值时, $k\in\overline{\mathbb{Z}}_+$, 则$\Delta V(x(k),\hat{x}(k))$是由(2) 给出的随机变量. 其中期望值是对$\omega$取的, 即$x=x(k)$, $\hat{x}=\hat{x}(k)$被视为常数. 实际上, $V_s(x(k))$ 是全概率空间$(\Omega,\mathcal{F},\mathcal{P})$ 中的元素, 于是$\Delta V_s(x(k),\hat{x}(k))=E[V_s(x(k+1),\hat{x}(k+1))|\mathcal{F}_k]-V_s(x(k),\hat{x}(k))$, 为了叙述方便, 我们记作(2) 的形式.

Willems 在 [4]中对非线性离散动力系统的耗散理论进行了详细研究. 据此, 我们给出储能函数和供给率, 并由此介绍非线性随机离散系统几何随机增量QSR耗散的定义, 严格几何随机增量QSR耗散的定义, 局部严格几何随机增量QSR耗散的充分的LMI条件, 以及随机增量QSR耗散与严格几何随机增量QSR耗散的关系.

引理 [4]对于系统(1), 假设存在一个连续函数V_s: $\mathcal{D}\to\mathbb{R}$, 使得

$$V_s(0,0)=0,$$

其中$V_s(x,\hat{x}) \geq 0$, x, $\hat{x} \in \mathbb{R}^n$, $x \neq \hat{x}$, 并且

$$E[V_s(x(k+1),\hat{x}(k+1))] = E[V_1(x(k+1))] + E[V_2(\hat{x}(k+1))], \tag{3}$$

$$E[V_1(f(x)+G(x)u+D(x)\omega)] = E[V_1(f(x)+D(x)\omega)] + P_{1u}(x)u + u^T P_{2u}(x)u, \tag{4}$$

$$E[V_2(f(\hat{x})+G(\hat{x})\hat{u})+D(\hat{x})\omega] = E[V_2(f(\hat{x})+D(\hat{x})\omega)] + P_{1\hat{u}}(\hat{x})\hat{u} + \hat{u}^T P_{2\hat{u}}(\hat{x})\hat{u}. \tag{5}$$

另外, 如果存在$\alpha > 0$, $\beta > \gamma > 0$, $C_3, C_4 \in \mathbb{R}_+$, 使得

$$\rho E[V_s(x(k+1),\hat{x}(k+1))] \leq V_s(x(k),\hat{x}(k)), \tag{6}$$

$$\alpha \parallel x-\hat{x} \parallel^2 \leq V_s(x,\hat{x}) \leq \beta \parallel x-\hat{x} \parallel^2, \tag{7}$$

$$\Delta V_s(x,\hat{x}) \leq -\gamma \parallel x-\hat{x} \parallel^2, \tag{8}$$

$$\|E[V_s^{'}(f(x)+D(x)\omega)]\| \leq C_3 \parallel x-\hat{x} \parallel, \tag{9}$$

$$\|E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]\| \leq C_4 \parallel x-\hat{x} \parallel, \tag{10}$$

则系统(1)为概率意义下的几何均方增量稳定. 下面的定义介绍了系统(1)的几何随机增量耗散的几个概念. 另外, 我们假设存在一个函数$\zeta : \mathbb{R}^l \to \mathbb{R}^l$, 使得$\zeta(0,0) = 0$并且$r(\zeta(y,\hat{y}),\Delta y) \geq 0$, $y \neq \hat{y}$. 并且$r(\Delta u,\Delta y)$是一个连续可微函数. 然后, 存在供给率$r(\Delta u,\Delta y)$: $\mathbb{U} \times Y \to \mathbb{R}$ 使得$r(0,0) = 0$, 且$r(\Delta u(k),\Delta y(k))$ 对于所有满足系统(1)的输入输出对是局部可和的. 也就是说, 对于所有的输入输出对$u(\cdot) \in \mathbb{U}$ 和$y(\cdot) \in Y$ 满足(1), 并且所有的有界停时$\kappa \geq 0$, $r(\cdot,\cdot)$ 满足

$$E[\sum_{i=k_0}^{\kappa-1} \mid r(\Delta u(i),\Delta y(i)) \mid] \geq 0, \kappa-1 \geq \kappa_0, \tag{11}$$

其中$\Delta u = u-\hat{u}, \Delta y = y-\hat{y}$.

定义1 [4]如果系统(1)是的几何随机增量QSR耗散的, 则存在一个非负定的储能函数$V_s(x,\hat{x})$, 满足$V_s(0,0) = 0$, 并且存在供给率$r(\Delta u,\Delta y)$ 使得

$$\rho E[V_s(x(k+1),\hat{x}(k+1))] - V_s(x(k),\hat{x}(k)) \leq \rho r(\Delta u(k),\Delta y(k)), \tag{12}$$

其中$\rho > 1$, $r(\Delta u,\Delta y) = \Delta y^T Q \Delta y + 2\Delta y^T S \Delta u + \Delta u^T R \Delta u$, $Q \in \mathbb{S}^l$, $R \in \mathbb{S}^m$, 和$S \in \mathbb{R}^{l\times m}$.

定义2 [4]对于随机增量QSR耗散的系统(1), 存在$T(x,\hat{x})>0$, 使得

$$E[V_s(x(k+1),\hat{x}(k+1))]-V_s(x(k),\hat{x}(k))+T(x,\hat{x})\leq \Delta y^TQ\Delta y+2\Delta y^TS\Delta u+\Delta u^TR\Delta u, \quad (13)$$

则称系统(1)为严格几何随机增量耗散的. 其中$Q\in\mathbb{S}^l$, $R\in\mathbb{S}^m$ 和$S\in\mathbb{R}^{l\times m}$.

下面给出一个$\mathcal{D}\times\mathbb{U}$中的对于局部严格几何随机增量QSR耗散的充分的LMI条件

$$\begin{pmatrix} \Phi & \frac{1}{2}P_{1u}(x)-\frac{1}{2}P_{1\hat{u}}(\hat{x})-\Delta h^T(x)S \\ (\frac{1}{2}P_{1u}(x)-\frac{1}{2}P_{1\hat{u}}(\hat{x})-\Delta h^T(x)S)^T & -R+P_{2u}(x)+P_{2\hat{u}}(\hat{x}) \end{pmatrix}\leq 0, \quad (14)$$

其中$\Phi=E[V_1(f(x)+D(x)\omega)]+E[V_2(f(\hat{x})+D(\hat{x})\omega)]-V_1(x(k))-V_2(\hat{x}(k))+T(x,\hat{x})-\Delta h^TQ\Delta h$. 对于所有的$x,\hat{x}\in\mathbb{R}^n$, $P_{1u}(x)$: $\mathbb{R}^n\rightarrow\mathbb{R}^{1\times m}$, $P_{2u}(x)$: $\mathbb{R}^n\rightarrow\mathbb{N}^m$,

$$P_{1u}(x)=\mathbb{E}[V_s^{'}(f(x)+D(x)w)]G(x), P_{1\hat{u}}(\hat{x})=\mathbb{E}[V_s^{'}(f(\hat{x})+D(\hat{x})w)]G(\hat{x})$$

$$P_{2u}(x)=\frac{1}{2}G^T(x)\mathbb{E}[V_s^{''}(f(x)+D(x)w)]G(x), P_{2\hat{u}}(\hat{x})=\frac{1}{2}G^T(\hat{x})\mathbb{E}[V_s^{''}(f(\hat{x})+D(\hat{x})w)]G(\hat{x}),$$

并且

$$P_{1u}(x)\hat{u}+P_{1\hat{u}}(\hat{x})u=0, \quad (15)$$

$$P_{2u}(x)\hat{u}=P_{2\hat{u}}(\hat{x})u=0. \quad (16)$$

定义3 [4]对于严格几何随机增量QSR耗散的动态系统(1), 存在$\rho>1$, 使得

$$T=\frac{\rho-1}{\rho}V_s(x,\hat{x})=\frac{\rho-1}{\rho}V_1(x)+\frac{\rho-1}{\rho}V_2(\hat{x}). \quad (17)$$

则称它为几何随机增量QSR耗散的.

3. 主要结果

接下来, 我们将推导离散随机非线性系统的线性静态输出反馈控制器. 为此, 可以设计线性静态输出反馈控制器$\Delta u=K\Delta y$, 利用几何随机增量QSR耗散实现概率下的的几何均方增量稳定性.

定理3.1 通过设计线性静态输出反馈控制器, 系统(1) 是概率意义下的几何均方增量稳定的当且仅当它是几何随机增量QSR耗散的, 满足$R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})>0$, $\Delta_1\leq 0$. 其中

$$\Delta_1=S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}-K^TP_{2u}(x)K-Q, \quad (18)$$

$$\Delta_2 = S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1} S^{\mathrm{T}} - K^T P_{2\hat{u}}(\hat{x}) K - Q. \tag{19}$$

设计的线性静态输出反馈控制器为

$$\Delta u = K\Delta y = K\Delta h(x), K = -[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1} S^T. \tag{20}$$

证明：必要性：由(7)和(8)可得

$$E[V_s(f(x) + D(x)\omega + G(x)u, f(\hat{x}) + D(\hat{x})\omega + G(\hat{x})\hat{u})] - V_s(x, \hat{x}) \le -\gamma \parallel x - \hat{x} \parallel^2 \le -\frac{\gamma}{\beta} V_s(x, \hat{x}), \tag{21}$$

移项可得

$$E[V_s(f(x) + D(x)\omega + G(x)u, f(\hat{x}) + D(\hat{x})\omega + G(\hat{x})\hat{u})] - (1 - \frac{\gamma}{\beta}) V_s(x, \hat{x}) \le 0. \tag{22}$$

令$\frac{1}{\rho} = \eta = 1 - \frac{\gamma}{\beta}$, 其中$0 < \eta < 1$ 从而

$$E[V_s(f(x) + D(x)\omega + G(x)u, f(\hat{x}) + D(\hat{x})\omega + G(\hat{x})\hat{u})] - \eta V_s(x, \hat{x}) \le 0, \tag{23}$$

令$\bar{\eta} > 0$, 于是

$$E[V_s(f(x) + D(x)\omega + G(x)u, f(\hat{x}) + D(\hat{x})\omega + G(\hat{x})\hat{u})] - \bar{\eta} V_s(x, \hat{x}) < 0. \tag{24}$$

其中$\frac{1}{\bar{\rho}} = \bar{\eta} = \frac{1+\varepsilon}{\rho}$, $0 < \varepsilon < \rho - 1$, 进而可得

$$E[V_s(f(x) + D(x)\omega + G(x)u, f(\hat{x}) + D(\hat{x})\omega + G(\hat{x})\hat{u})] - \bar{\eta} V_s(x, \hat{x}) \le -\frac{\varepsilon}{\rho} V_s(x, \hat{x}) < 0. \tag{25}$$

对于函数$V_s \in L^1$, 存在$\theta \in \overline{\mathbb{R}}_+$, 有

$$\begin{aligned} &E[V_s(f(x) + D(x)\omega + G(x)u, f(\hat{x}) + D(\hat{x})\omega + G(\hat{x})\hat{u})] \\ &-\bar{\eta} V_s(x, \hat{x}) + \frac{\theta}{4} (P_{1u}(x) - P_{1\hat{u}}(\hat{x}))(P_{1u}(x) - P_{1\hat{u}}(\hat{x}))^T \le 0, \end{aligned} \tag{26}$$

或者

$$\begin{aligned} &E[V_s(f(x) + D(x)\omega + G(x)u, f(\hat{x}) + D(\hat{x})\omega + G(\hat{x})\hat{u})] \\ &-\bar{\eta} V_s(x, \hat{x}) \le -\frac{\theta}{4} (P_{1u}(x) - P_{1\hat{u}}(\hat{x}))(P_{1u}(x) - P_{1\hat{u}}(\hat{x}))^T. \end{aligned} \tag{27}$$

通过在(26)中使用迹算子, 我们有

$$
\begin{aligned}
& E[V_s(f(x)+D(x)\omega+G(x)u, f(\hat{x})+D(\hat{x})\omega+G(\hat{x})\hat{u})] \\
& -\bar{\eta}V_s(x,\hat{x})+\frac{\theta}{4}(P_{1u}(x)-P_{1\hat{u}}(\hat{x}))(P_{1u}(x)-P_{1\hat{u}}(\hat{x}))^T \\
=& E[V_s(f(x)+D(x)\omega+G(x)u, f(\hat{x})+D(\hat{x})\omega+G(\hat{x})\hat{u})] \\
& -\bar{\eta}V_s(x,\hat{x})+\frac{\theta}{4}tr[(P_{1u}(x)-P_{1\hat{u}}(\hat{x}))(P_{1u}(x)-P_{1\hat{u}}(\hat{x}))^T] \\
\leq & -\frac{\varepsilon}{\rho}V_s(x,\hat{x})+\frac{\theta}{4}tr[(E[V_s^{'}(f(x)+D(x)\omega)]G(x) \\
& -E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]G(\hat{x}))(E[V_s^{'}(f(x)+D(x)\omega)]G(x)-E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]G(\hat{x})^T)] \\
\leq & -\frac{\varepsilon}{\rho}V_s(x,\hat{x})+\frac{\theta}{4}tr[(E[V_s^{'}(f(x)+D(x)\omega)]G(x))(E[V_s^{'}(f(x)+D(x)\omega)]G(x))^T \\
& -(E[V_s^{'}(f(x)+D(x)\omega)]G(x))(E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]G(\hat{x})^T) \\
& -(E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]G(\hat{x}))(E[V_s^{'}(f(x)+D(x)\omega)]G(x))^T \\
& +(E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]G(\hat{x}))(E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]G(\hat{x})^T)] \\
\leq & -\frac{\varepsilon}{\rho}V_s(x,\hat{x})+\frac{\theta}{4}[tr(G(x)G^T(x))(E[V_s^{'}(f(x)+D(x)\omega)]E[V_s^{'}(f(x)+D(x)\omega)]^T) \\
& -tr(G(x)G^T(\hat{x})(E[V_s^{'}(f(x)+D(x)\omega)]E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]^T) \\
& -tr(G(\hat{x})G^T(x))(E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]E[V_s^{'}(f(x)+D(x)\omega)]^T) \\
& +tr(G(\hat{x})G^T(\hat{x}))(E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]^T)],
\end{aligned}
\tag{28}
$$

这里, 实函数$tr(G(x)^TG(x))\geq 0$, 因此, 存在$\mathcal{G}\in\overline{\mathbb{R}}_+$, 使得$tr(G(x)G^T(x))\leq\mathcal{G}$, $tr(G(x)G^T(\hat{x}))\leq\mathcal{G}$, $tr(G(\hat{x})G^T(\hat{x}))\leq\mathcal{G}$. 与此同时, 通过(7), (10)和(11), 我们得到

$$
\begin{aligned}
& -\frac{\varepsilon}{\rho}V_s(x,\hat{x})+\frac{\theta}{4}\mathcal{G}[(E[V_s^{'}(f(x)+D(x)\omega)]E[V_s^{'}(f(x)+D(x)\omega)]^T) \\
& -2(E[V_s^{'}(f(x)+D(x)\omega)]E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]^T) \\
& +(E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]E[V_s^{'}(f(\hat{x})+D(\hat{x})\omega)]^T)] \\
\leq & -\frac{\varepsilon}{\rho}\alpha\parallel x-\hat{x}\parallel^2+\frac{\theta}{4}\mathcal{G}(C_3^2\parallel x-\hat{x}\parallel^2-2C_3C_4\parallel x-\hat{x}\parallel^2+C_4^2\parallel x-\hat{x}\parallel^2).
\end{aligned}
\tag{29}
$$

不等式(29)是非正的, 如果

$$-\frac{\varepsilon}{\rho}\alpha+\frac{\theta}{4}\mathcal{G}(C_3^2-2C_3C_4+C_4^2)\le 0, \tag{30}$$

等价于

$$0<\theta\le\frac{4\alpha\varepsilon}{\rho\mathcal{G}(C_3-C_4)^2}. \tag{31}$$

因此, 存在$\theta\in\overline{\mathbb{R}}_+$, 使得(26)和(27)是可行的. 假设(7)和(8)满足严格的几何随机增量QSR耗散, 并且$T=\frac{\overline{\rho}-1}{\overline{\rho}}V_s(x,\hat{x})$. 对于$R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})>0$, (14)是可行的, 当且仅当

$$\begin{aligned}&E[V_1(f(x)+D(x)\omega)]+E[V_2(f(\hat{x})+D(\hat{x})\omega)]-V_1(x)-V_2(\hat{x})+T\\&-[h(x)-h(\hat{x})]^{\mathrm{T}}Q[h(x)-h(\hat{x})]+(\frac{1}{2}P_{1u}(x)-\frac{1}{2}P_{1u}(\hat{x})-h^{\mathrm{T}}(x)S+h^{\mathrm{T}}(\hat{x})S)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}\\&\times(\frac{1}{2}P_{1u}(x)-\frac{1}{2}P_{1u}(\hat{x})-h^{\mathrm{T}}(x)S+h^{\mathrm{T}}(\hat{x})S)^{\mathrm{T}}\le 0,\end{aligned} \tag{32}$$

或者

$$\begin{aligned}&E[V_1(f(x)+D(x)\omega)]+E[V_2(f(\hat{x})+D(\hat{x})\omega)]-V_1(x)-V_2(\hat{x})+T-[h^{\mathrm{T}}(x)Qh(x)-h^{\mathrm{T}}(x)Qh(\hat{x})\\&-h^{\mathrm{T}}(\hat{x})Qh(x)+h^{\mathrm{T}}(\hat{x})Qh^{\mathrm{T}}(\hat{x})]\\&+(\frac{1}{2}P_{1u}(x)-\frac{1}{2}P_{1u}(\hat{x})-h^{\mathrm{T}}(x)S+h^{\mathrm{T}}(\hat{x})S)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}\\&\times(\frac{1}{2}P_{1u}(x)-\frac{1}{2}P_{1u}(\hat{x})-h^{\mathrm{T}}(x)S+h^{\mathrm{T}}(\hat{x})S)^{\mathrm{T}}\le 0,\end{aligned} \tag{33}$$

进一步展开可得

$$\begin{aligned}&E[V_1(f(x)+D(x)\omega)]+E[V_2(f(\hat{x})+D(\hat{x})\omega)]-V_1(x)-V_2(\hat{x})+T-h^{\mathrm{T}}(x)Qh(x)+2h^{\mathrm{T}}(x)Qh(\hat{x})\\&-h^{\mathrm{T}}(\hat{x})Qh(\hat{x})+\frac{1}{4}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)-\frac{1}{4}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\&-\frac{1}{2}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(x)+\frac{1}{2}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x})\\&-\frac{1}{4}P_{1\hat{u}}(\hat{x})[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)+\frac{1}{4}P_{1\hat{u}}(\hat{x})[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\&+\frac{1}{2}P_{1\hat{u}}(\hat{x})[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(x)-\frac{1}{2}P_{1\hat{u}}(\hat{x})[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x})\\&-\frac{1}{2}h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)+\frac{1}{2}h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\&+h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(x)-h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x})\\&+\frac{1}{2}h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)-\frac{1}{2}h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\&-h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(x)+h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x})\le 0,\end{aligned} \tag{34}$$

化简可得

$$
\begin{aligned}
&E[V_1(f(x)+D(x)\omega)]+E[V_2(f(\hat{x})+D(\hat{x})\omega)]-V_1(x)-V_2(\hat{x})+T-h^{\mathrm{T}}(x)Qh(x)+2h^{\mathrm{T}}(x)Qh(\hat{x})\\
&-h^{\mathrm{T}}(\hat{x})Qh(\hat{x})+\frac{1}{4}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)-\frac{1}{2}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\
&-h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)+h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)\\
&+\frac{1}{4}P_{1\hat{u}}(\hat{x})[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})+h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\
&-h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})+h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(x)\\
&-2h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x})+h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x})\leq 0.
\end{aligned}
\tag{35}
$$

由(3), (4) 和(5), 如果$T=(1-\bar{\eta})=\frac{\bar{\rho}-1}{\bar{\rho}}V_s(x,\hat{x})$, 使得

$$
\begin{aligned}
&-P_{1u}(x)u-u^{\mathrm{T}}P_{2u}(x)u-P_{1\hat{u}}(\hat{x})\hat{u}-\hat{u}^{\mathrm{T}}P_{2\hat{u}}(\hat{x})\hat{u}-\frac{\theta}{4}[P_{1u}(x)-P_{1\hat{u}}(\hat{x})][P_{1u}(x)-P_{1\hat{u}}(\hat{x})]^{\mathrm{T}}\\
\leq& h^{T}(x)Qh(x)-2h^{\mathrm{T}}(x)Qh(\hat{x})+h^{\mathrm{T}}(\hat{x})Qh(\hat{x})-\frac{1}{4}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)\\
&+\frac{1}{2}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})+h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)\\
&-h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)-\frac{1}{4}P_{1\hat{u}}(\hat{x})[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\
&-h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})+h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\
&-h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(x)+2h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x})\\
&-h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x}),
\end{aligned}
\tag{36}
$$

因为$R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})=\frac{I}{\theta}>0$, 其中$\theta$来自(26), 并且$I$ 为单位矩阵, 从而

$$
\begin{aligned}
&-P_{1u}(x)u-u^{\mathrm{T}}P_{2u}(x)u-P_{1\hat{u}}(\hat{x})\hat{u}-\hat{u}^{\mathrm{T}}P_{2\hat{u}}(\hat{x})\hat{u}\\
&-\frac{1}{4}[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}(x)P_{1u}^{\mathrm{T}}(x)+\frac{1}{2}[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}(x)P_{1\hat{u}}^{\mathrm{T}}(\hat{x})\\
&-\frac{1}{4}[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}(\hat{x})P_{1\hat{u}}^{\mathrm{T}}(\hat{x})\\
\leq& h^{T}(x)Qh(x)-2h^{\mathrm{T}}(x)Qh(\hat{x})+h^{\mathrm{T}}(\hat{x})Qh(\hat{x})-\frac{1}{4}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)\\
&+\frac{1}{2}P_{1u}(x)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})+h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)\\
&-h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x)-\frac{1}{4}P_{1\hat{u}}(\hat{x})[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\
&-h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})+h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x})\\
&-h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(x)+2h^{\mathrm{T}}(x)S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x})\\
&-h^{\mathrm{T}}(\hat{x})S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x}).
\end{aligned}
\tag{37}
$$

又由于$\Delta u = K\Delta h(x)$, 进而

$$
\begin{aligned}
&-P_{1u}(x)Kh(x) - h^{\mathrm{T}}(x)K^{\mathrm{T}}P_{2u}(x)Kh(x) - P_{1\hat{u}}(\hat{x})Kh(\hat{x}) - h^{\mathrm{T}}(\hat{x})K^{\mathrm{T}}P_{2\hat{u}}(\hat{x})Kh(\hat{x}) \\
\leq& h^{T}(x)Qh(x) - 2h^{\mathrm{T}}(x)Qh(\hat{x}) + h^{\mathrm{T}}(\hat{x})Qh(\hat{x}) + h^{\mathrm{T}}(x)S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x) \\
&-h^{\mathrm{T}}(\hat{x})S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x) - h^{\mathrm{T}}(x)S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x}) \\
&+h^{\mathrm{T}}(\hat{x})S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x}) - h^{\mathrm{T}}(x)S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(x) \\
&+2h^{\mathrm{T}}(x)S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x}) - h^{\mathrm{T}}(\hat{x})S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}h(\hat{x}),
\end{aligned} \tag{38}
$$

进一步化简可得

$$
\begin{aligned}
&h^{\mathrm{T}}(x)[Q - S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}} + K^{T}P_{2u}(x)K]h(x) \\
&+h^{\mathrm{T}}(\hat{x})[Q - S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}} \\
&+K^{T}P_{2\hat{u}}(\hat{x})K]h(\hat{x}) + h^{\mathrm{T}}(x)[K^{T} + S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}]P_{1u}(x) \\
&+h^{\mathrm{T}}(\hat{x})[K^{T} + S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}]P_{1\hat{u}}(\hat{x}) \\
&+2h^{\mathrm{T}}(x)[-Q + S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}]h^{\mathrm{T}}(\hat{x}) - h^{\mathrm{T}}(\hat{x})S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}P_{1u}^{T}(x) \\
&-h^{\mathrm{T}}(x)S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}P_{1\hat{u}}^{T}(\hat{x}) \geq 0,
\end{aligned} \tag{39}
$$

通过$R - P_{2u}(x) - P_{2\hat{u}}(\hat{x}) = \frac{I}{\theta}$和(39), 我们有

$$
-\Delta_1 = Q - S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}} + K^{T}P_{2u}(x)K, \tag{40}
$$

$$
-\Delta_2 = Q - S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}} + K^{T}P_{2\hat{u}}(\hat{x})K. \tag{41}
$$

对于$R - P_{2u}(x) - P_{2\hat{u}}(\hat{x}) = \frac{I}{\theta}$, $K = -[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{T}$, 并且由等式(15), (39)可化为

$$
-h^{\mathrm{T}}(x)\Delta_1 h(x) - h^{\mathrm{T}}(\hat{x})\Delta_2 h(\hat{x}) + 2h^{\mathrm{T}}(x)[-Q + S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}}]h(\hat{x}) \geq 0, \tag{42}
$$

进一步地

$$
\begin{aligned}
&-h^{\mathrm{T}}(x)\Delta_1 h(x) - h^{\mathrm{T}}(\hat{x})\Delta_2 h(\hat{x}) \\
&+2h^{\mathrm{T}}(x)[-Q + S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}} - K^{T}P_{2u}(x)K]h(\hat{x}) \geq 0,
\end{aligned} \tag{43}
$$

等价于

$$
\begin{aligned}
&-h^{\mathrm{T}}(x)\Delta_1 h(x) - h^{\mathrm{T}}(\hat{x})\Delta_2 h(\hat{x}) + h^{\mathrm{T}}(x)[-Q + S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}} - K^{T}P_{2u}(x)K]h(\hat{x}) \\
&+h^{\mathrm{T}}(x)[-Q + S[R - P_{2u}(x) - P_{2\hat{u}}(\hat{x})]^{-1}S^{\mathrm{T}} - K^{T}P_{2\hat{u}}(\hat{x})K]h(\hat{x}) \geq 0,
\end{aligned} \tag{44}
$$

考虑等式(16), 于是

$$
\begin{aligned}
&-h^{\mathrm{T}}(x)\Delta_1 h(x)-h^{\mathrm{T}}(\hat{x})\Delta_2 h(\hat{x})+h^{\mathrm{T}}(x)\Delta_1 h(\hat{x})+h^{\mathrm{T}}(x)\Delta_2 h(\hat{x})\\
=&-h^{\mathrm{T}}(x)\Delta_1[h(x)-h(\hat{x})]+[h^{\mathrm{T}}(x)-h^{\mathrm{T}}(\hat{x})]\Delta_2 h(\hat{x})\\
=&-[h^{\mathrm{T}}(x)\Delta_1-h^{\mathrm{T}}(\hat{x})\Delta_2][h(x)-h(\hat{x})]\\
=&-[h^{\mathrm{T}}(x)\Delta_1-h^{\mathrm{T}}(\hat{x})\Delta_1+h^{\mathrm{T}}(\hat{x})\Delta_1-h^{\mathrm{T}}(\hat{x})\Delta_2][h(x)-h(\hat{x})]\\
=&-[[h^{\mathrm{T}}(x)-h^{\mathrm{T}}(\hat{x})]\Delta_1+h^{\mathrm{T}}(\hat{x})[K^T P_{2u}(x)K-K^T P_{2\hat{u}}(\hat{x})K]]\Delta h(x)\\
=&-\Delta h^{\mathrm{T}}(x)\Delta_1\Delta h(x)+h^{\mathrm{T}}(\hat{x})K^T P_{2\hat{u}}(\hat{x})Kh(\hat{x})\geq 0,
\end{aligned}
\tag{45}
$$

最终可得

$$
\Delta_1\leq 0. \tag{46}
$$

因此, 有了$R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})>0$, $\Delta_1\leq 0$, 系统(1)是几何随机增量QSR耗散的.

充分性: 假设系统(1)是几何随机增量QSR耗散的, 通过$T=\frac{\rho-1}{\rho}V_s(x,\hat{x})$, 然后存在供给率$r(\Delta u(k),\Delta y(k))=\Delta y^{\mathrm{T}}Q\Delta y+2\Delta y^{\mathrm{T}}S\Delta u+\Delta u^{\mathrm{T}}R\Delta u$, 使得

$$
\rho E[V_s(f(x)+G(x)u+D(x)\omega,f(\hat{x})+G(\hat{x})\hat{u}+D(\hat{x})\omega]-V_s(x,\hat{x})\leq\rho r(\Delta u(k),\Delta y(k)), \tag{47}
$$

等价于

$$
E[V_s(f(x)+G(x)u+D(x)\omega,f(\hat{x})+G(\hat{x})\hat{u}+D(\hat{x})\omega]-\frac{1}{\rho}V_s(x,\hat{x})\leq r(\Delta u(k),\Delta y(k)), \tag{48}
$$

对于$\Delta u=K\Delta y$, 从而

$$
r(K\Delta y,\Delta y)=\Delta y^{\mathrm{T}}Q\Delta y+2\Delta y^{\mathrm{T}}SK\Delta y+\Delta y^{\mathrm{T}}K^{\mathrm{T}}RK\Delta y. \tag{49}
$$

由于供给率$r(\Delta u(k),\Delta y(k))$是连续的, 进而

$$
\frac{dr}{d\Delta y}=2\Delta yQ+4SK\Delta y+2K^{\mathrm{T}}RK\Delta y=2\Delta y(Q+2SK+K^{\mathrm{T}}RK), \tag{50}
$$

其中$K=-[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T$, 考虑

$$
\begin{aligned}
&Q+2SK+K^T RK\\
=&Q-2S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T\\
&+S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}R[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T\\
=&Q-2S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})][R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T
\end{aligned}
$$

$$
\begin{aligned}
&+S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}R[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T\\
&=Q-S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T\\
&+S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}[P_{2u}(x)+P_{2\hat{u}}(\hat{x})][R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T\\
&=Q-S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T+K^T[P_{2u}(x)+P_{2\hat{u}}(\hat{x})]K\\
&=-\Delta_1+K^TP_{2\hat{u}}(\hat{x})K,
\end{aligned}
\tag{51}
$$

由(51), (50)可化简为

$$
\frac{dr}{d\Delta y}=2\Delta y[-\Delta_1+K^TP_{2\hat{u}}(\hat{x})K], \tag{52}
$$

由于$\Delta_1\leq 0$, 进而可得

$$
\frac{d^2r}{d\Delta y^2}=2[-\Delta_1+K^TP_{2\hat{u}}(\hat{x})K]\geq 0, \tag{53}
$$

即供给率函数$r(\Delta u,\Delta y)$有最小值. 于是, 我们令

$$
\frac{dr}{d\Delta y}=2\Delta y[-\Delta_1+K^TP_{2\hat{u}}(\hat{x})K]=0, \tag{54}
$$

当且仅当

$$
\Delta y=0, \tag{55}
$$

从而可得

$$
r(K\Delta y,\Delta y)=\Delta y^{\mathrm{T}}Q\Delta y+2\Delta y^{\mathrm{T}}SK\Delta y+\Delta y^{\mathrm{T}}K^{\mathrm{T}}RK\Delta y=0, \tag{56}
$$

也就是说

$$
\min r(\Delta u,\Delta y)=0, \tag{57}
$$

由(47)可得

$$
E[V_s(f(x)+G(x)u+D(x)\omega,f(\hat{x})+G(\hat{x})\hat{u}+D(\hat{x})\omega]-V_s(x,\hat{x})\leq(\frac{1}{\rho}-1)V_s(x,\hat{x})\leq(\frac{1}{\rho}-1)\alpha||x-\hat{x}||^2, \tag{58}
$$

通过(58)和(8), 我们有$-\gamma=(\frac{1}{\rho}-1)\alpha$, 这和(47)是一致的. 因此, 原点具有概率意义下的几何均方增量稳定性, 并且当$\Delta_1\leq 0$时满足稳定的充分条件。

注: 通过设计线性静态输出反馈控制器$\Delta u=K\Delta h$, 非线性随机动态系统(1)是概率意义下的几何均方增量稳定的当且仅当它是几何随机增量QSR耗散的, 满足$R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})>0$, $\Delta_1\leq 0$. 到此, 我们证明了定理3.1成立.

4. 数值模拟

在本节中, 我们将提供一个示例来说明增量控制器的有效性. 考虑下面的非线性随机离散动态系统

$$\begin{aligned}&x_1(k+1)=x_1(k)\omega(k),x_1(0)=x_{10},k\in\mathbb{Z}^+,\\&x_2(k+1)=\frac{1}{2}x_1^2(k)+u(k),x_2(0)=x_{20},\\&y(k)=2x_1^2(k),\end{aligned}\tag{59}$$

其中$\omega(k),k\geq 0$, 是均值为零, 标准差为σ的独立高斯随机变量序列. 对于$x=[x_1,x_2]^T$, $\hat{x}=[\hat{x_1},\hat{x_2}]^T$, $f(x)=[0,\frac{1}{2}x_1^2]^T$, $f(\hat{x})=[0,\frac{1}{2}\hat{x}_1^2]^T$, $G(x)=[0,1]^T,G(\hat{x})=[0,1]^T$, $h(x)=2x_1^2,h(\hat{x})=2\hat{x}_1^2$, $D(x)=[x_1,0]^T$, $D(\hat{x})=[\hat{x}_1,0]^T$, 给定储能函数$V_s(x)=x_1^4+8x_2^2,V_s(\hat{x})=\hat{x}_1^4+8\hat{x}_2^2,V_s(x,\hat{x})=x_1^4+8x_2^2+\hat{x}_1^4+8\hat{x}_2^2$ 由(15)和(16), 设置$P_{1u}(x)=0,P_{1\hat{u}}(\hat{x})=0$, $P_{2u}(x)=0,P_{2\hat{u}}(\hat{x})=0$, 根据需要, 我们给定$Q=3$, $R=1$, 和$S=I$. 于是, 我们给出供给率函数$r(\Delta u,\Delta y)=3\Delta y\Delta y^T+2\Delta y^T\Delta u+\Delta u\Delta u^T$. 进而$K=-[R-P_{2u}(x)-P_{2\hat{u}}]^{-1}S^T=-I$.

通过计算可得

$$\Delta u=K\Delta y=-(2x_1^2-2\hat{x}_1^2),\tag{60}$$

$$\begin{aligned}r(\Delta u,\Delta y)&=3(2x_1^2-2\hat{x}_1^2)^2-2(2x_1^2-2\hat{x}_1^2)^2+(2x_1^2-2\hat{x}_1^2)^2\\&=2(2x_1^2-2\hat{x}_1^2)^2\\&=8(x_1^2-\hat{x}_1^2)^2\geq 0.\end{aligned}\tag{61}$$

进一步地, 限定$1\leq x_1\leq x_2\leq\sqrt{2}$, $1\leq\hat{x}_1\leq\hat{x}_2\leq\sqrt{2}$, 从而$x_1^2\leq x_1^4\leq 2x_1^2\leq 2x_2^2$, $\hat{x}_1^2\leq\hat{x}_1^4\leq 2\hat{x}_1^2\leq 2\hat{x}_2^2$. 于是, 对于$V_s(0,0)=0$, $V_s(x,\hat{x})>0$, 并且$x\neq\hat{x}$, 我们有

$$\begin{aligned}\Delta V_s(x,\hat{x})&=E[V_s(f(x)+G(x)u+D(x)\omega,f(\hat{x})+G(\hat{x})\hat{u}+D(\hat{x})\omega]-V_s(x,\hat{x})\\&=E[V_s(f(x)+G(x)u+D(x)\omega,f(\hat{x})+G(\hat{x})\hat{u}+D(\hat{x})\omega]-V_s(x)-V_s(\hat{x})\\&=E[(\omega x_1)^4+2x_1^4]+E[(\omega\hat{x}_1)^4+2x_1^4]-x_1^4-\hat{x}_1^4-8(x_2^2+\hat{x}_2^2)\\&=(3\sigma^4+1)(x_1^4+\hat{x}_1^4)-8(x_2^2+\hat{x}_2^2)\\&\leq(6\sigma^4+2)(x_2^2+\hat{x}_2^2)-8(x_2^2+\hat{x}_2^2)\leq r(\Delta u,\Delta y)=8(x_1^2-\hat{x}_1^2)^2.\end{aligned}\tag{62}$$

由(62), $6\sigma^4-6\leq 0$, $\sigma^4\leq 1$. 这就得出了σ的取值范围.

$$x_1^2+x_2^2+\hat{x}_1^2+\hat{x}_2^2\leq V_s(x,\hat{x})=x_1^4+8x_2^2+\hat{x}_1^4+8\hat{x}_2^2\leq 2x_1^2+8x_2^2+2\hat{x}_1^2+8\hat{x}_2^2\leq 8(x_1^2+x_2^2+\hat{x}_1^2+\hat{x}_2^2).\tag{63}$$

也就是说$\alpha=1$, $\beta=8$, 从而验证了(7)成立.

取$\sigma^4=\frac{1}{12}<1$, 于是

$$\begin{aligned}\Delta V_s(x,\hat{x}) &= E[V_s(f(x)+G(x)u+D(x)\omega, f(\hat{x})+G(\hat{x})\hat{u}+D(\hat{x})\omega]-V_s(x,\hat{x}) \\ &= (6\sigma^4-6)(x_2^2+\hat{x}_2^2) \\ &= -4(x_2^2+\hat{x}_2^2) \\ &= -2(x_2^2+\hat{x}_2^2)-2(x_2^2+\hat{x}_2^2) \\ &< -2(x_1^2+\hat{x}_1^2)-2(x_2^2+\hat{x}_2^2) \\ &= -2(x_1^2+\hat{x}_1^2+x_2^2+\hat{x}_2^2). \end{aligned} \tag{64}$$

因此$\gamma=2<\beta$, 则(8)成立. 因此, 非线性系统(59)满足概率意义上的几何均方增量稳定. 并且, $\frac{1}{\rho}=1-\frac{\gamma}{\beta}=\frac{3}{4}$

通过计算可得$V_s'(x)=4x_1^3+16x_2$, $f(x)+D(x)\omega=(\omega x_1,\frac{1}{2}x_1^2)^T$, $V_s'(f(x)+D(x)\omega)=4\omega^3x_1^3+8x_1^2$, 进而我们得出

$$E[V_s'(f(x)+D(x)\omega)]=8x_1^2\leq C_3x_1\leq C_3\sqrt{x_1^2+\hat{x}_1^2}. \tag{65}$$

由(65), 我们有$C_3\geq 8x_1$, 取$C_3=8\sqrt{2}$. 也就是说(10)成立. C_4同理, 且$C_3\neq C_4$.

接下来, 我们讨论(14)是否成立. 对于

$$\begin{aligned}&\Delta V_s(x,\hat{x})+T(x,\hat{x})-\Delta h^TQ\Delta h \\ =&\frac{9}{4}(x_1^4+\hat{x}_1^4)-\frac{3}{4}(x_1^4+8x_2^2)-\frac{3}{4}(\hat{x}_1^4+8\hat{x}_2^2)-3(2x_1^2-2\hat{x}_1^2)^2 \\ =&\frac{3}{2}(x_1^4+\hat{x}_1^4)-6(x_2^2+\hat{x}_2^2)-3(2x_1^2-2\hat{x}_1^2)^2 \\ \leq&3(x_2^2+\hat{x}_2^2)-6(x_2^2+\hat{x}_2^2)-3(2x_1^2-2\hat{x}_1^2)^2 \\ \leq&-3(x_2^2+\hat{x}_2^2)-3(2x_1^2-2\hat{x}_1^2)^2 \\ <&-3(2x_1^2-2\hat{x}_1^2)^2 \\ <&0, \end{aligned} \tag{66}$$

$$-R+P_{2u}(x)+P_{2\hat{u}}(\hat{x})=-1<0, \tag{67}$$

$$\Delta h^TS=2x_1^2-2\hat{x}_1^2, \tag{68}$$

$$\begin{aligned}\Delta =&S[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}S^T-K^TP_{2u}(x)K-Q \\ =&-2I<0, \end{aligned} \tag{69}$$

进而可得

$$
\begin{aligned}
&E[V_1(f(x)+D(x)\omega)]+E[V_2(f(\hat{x})+D(\hat{x})\omega)]-V_1(x)-V_2(\hat{x})+T-[h(x)-h(\hat{x})]^{\mathrm{T}}(x)Q[h(x)-h(\hat{x})]\\
&+(\frac{1}{2}P_{1u}(x)-\frac{1}{2}P_{1u}(\hat{x})-h^{\mathrm{T}}(x)S+h^{\mathrm{T}}(\hat{x})S)[R-P_{2u}(x)-P_{2\hat{u}}(\hat{x})]^{-1}\\
&\times(\frac{1}{2}P_{1u}(x)-\frac{1}{2}P_{1u}(\hat{x})-h^{\mathrm{T}}(x)S+h^{\mathrm{T}}(\hat{x})S)^{\mathrm{T}}\\
&<-3(2x_1^2-2\hat{x}_1^2)^2+(2x_1^2-2\hat{x}_1^2)^2\\
&<-(2x_1^2-2\hat{x}_1^2)^2\\
&<0.
\end{aligned}\tag{70}
$$

我们验证了条件(14)对于定理3.1成立. 因此, 根据定理3.1, 非线性随机离散系统(59)是概率意义下几何均方增量稳定的. 通过设置初值$x_{10}=1.0$, $\hat{x}_{10}=0.8$, $x_{20}=-1.0$, $\hat{x}_{20}=-0.8$. 图 1展示了在线性静态输出反馈控制器Δu下, 闭环系统(59)的状态. 此外, 图 1 表明线性静态输出反馈控制器Δu可以使闭环系统(59)稳定, 并且阐明了所提供的供给率函数的有效性.

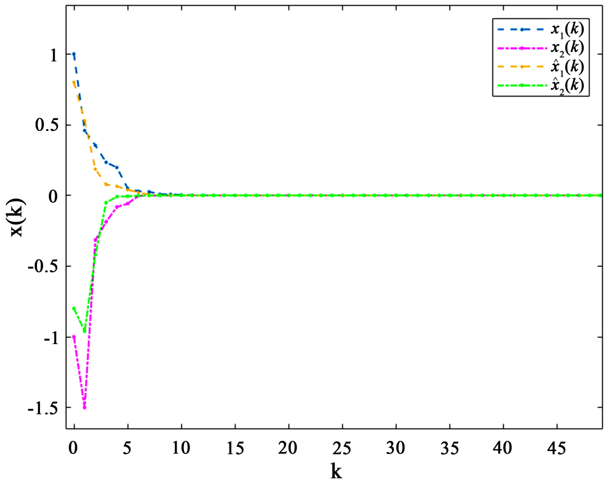

Figure 1. Trajectories of the state $x_1(k), x_2(k), \hat{x}_1(k), \hat{x}_2(k)$

图 1. 状态$x_1(k), x_2(k), \hat{x}_1(k), \hat{x}_2(k)$的轨线

综上所述, 通过数值模拟我们验证了所给储能函数和供给率函数的可行性, 并且设计的线性静态输出反馈控制器可以使系统(59)达到概率意义下几何均方增量稳定. 同时验证了(7), (8)以及(10)对于系统(59)成立, 最后通过具体的线性矩阵不等式验证了系统(59)的几何随机增量QSR耗散性.

5. 结论

本文的核心是研究非线性随机离散系统的概率意义下的几何均方增量稳定的充要条件. 首先, 利用设计的线性静态输出反馈控制器, 深入研究了非线性随机离散系统的几何均方增量稳定性问题. 接着给出了线性静态输出反馈控制器实现非线性随机离散系统的几何增量稳定性相当于表现出几

何随机增量QSR耗散性. 最后, 通过数值仿真验证了该方法的有效性. 在未来的研究中可以考虑把所得结论推广到线性变参数随机系统, 研究该系统的线性分数表示, 开拓新的研究方向.

基金项目

河北省自然科学基金项目(No. A2021204004; A2022204001); 河北农业大学大学生创新创业训练计划基金项目(No. 2024129; No. 2024126; No. 2024127); 河北农业大学人才引进项目(No. YJ2020036; PY2021003).

参考文献

[1] Brogliato, B., Lozano, R., Maschke, B. and Egeland, O. (2007) Dissipative Systems Analysis and Control. *Theory and Applications*, **2**, 2-5. https://doi.org/10.1007/978-1-84628-517-2_1

[2] Willems, J.C. (1972) Dissipative Dynamical Systems Part I: General Theory. *Archive for Rational Mechanics and Analysis*, **45**, 321-351. https://doi.org/10.1007/BF00276493

[3] Willems, J.C. (1972) Dissipative Dynamical Systems Part II: Linear Systems with Quadratic Supply Rates. *Archive for Rational Mechanics and Analysis*, **45**, 352-393. https://doi.org/10.1007/BF00276494

[4] Haddad, W. and Chellaboina, V. (2008) Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach. Princeton University Press, Princeton. https://doi.org/10.1515/9781400841042

[5] Rajpurohit, T. and Haddad, W.M. (2016) Dissipativity Theory for Nonlinear Stochastic Dynamical Systems. *IEEE Transactions on Automatic Control*, **62**, 1684-1699. https://doi.org/10.1109/TAC.2016.2598474

[6] Berman, N. and Shaked, U. (2006) H/sub/spl infin//Control for Discrete-Time Nonlinear Stochastic Systems. *IEEE Transactions on Automatic Control*, **51**, 1041-1046. https://doi.org/10.1109/TAC.2006.876808

[7] Liu, T., Hill, D.J. and Zhao, J. (2015) Output Synchronization of Dynamical Networks with Incrementally-Dissipative Nodes and Switching Topology. *IEEE Transactions on Circuits and Systems I: Regular Papers*, **62**, 2312-2323. https://doi.org/10.1109/TCSI.2015.2451891

[8] Aghannan, N. and Rouchon, P. (2003) An Intrinsic Observer for a Class of Lagrangian Systems. *IEEE Transactions on Automatic Control*, **48**, 936-945. https://doi.org/10.1109/TAC.2003.812778

[9] Richter, J.H., Heemels, W.P.M.H., van de Wouw, N. and Lunze, J. (2011) Reconfigurable Control of Piecewise Affine Systems with Actuator and Sensor Faults: Stability and Tracking. *Automatica*, **47**, 678-691. https://doi.org/10.1016/j.automatica.2011.01.048

[10] Madeira, D.D.S. (2021). Necessary and Sufficient Dissipativity-Based Conditions for Feedback Stabilization. *IEEE Transactions on Automatic Control*, **67**, 2100-2107. https://doi.org/10.1109/TAC.2021.3074850