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具有不规范非线性项的混合分数阶薛定谔方程组全局解的不存在性

蒲 俊

兰州理工大学，应用数学系，甘肃 兰州

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摘 要

本文研究了一个具有不规范非线性项的混合分数阶薛定谔方程组的柯西问题。通过引入有效的测试函数，导出关于解的加权积分的常微分不等式，利用常微分方程的性质证明了解会在有限时间内爆破，并得到了解存在时间的上估计。

关键词

测试函数，有限时间爆破，存在时间估计

Nonexistence of Global Solutions for Mixed Fractional Schrödinger Equations with Non-gauge Nonlinearities

Jun Pu

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou Gansu

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Abstract

This paper studies the Cauchy problem for a system of mixed fractional Schrödinger equations with non-gauge nonlinearities. By introducing an effective test function, we derive an ordinary differential inequality on the weighted integral of the solution, prove that the solution will blow-up in finite time by using the properties of ordinary differential equations, and obtain an upper estimate of the lifespan of the solution.

Keywords

Test Function, Finite Time Blow-Up, Lifespan

1. 引言

非线性薛定谔方程组(NLSEs)是一种常见的数学物理模型, 常用于描述许多复杂物理场中粒子波的相互作用与演化(见 [1–6]). 本文中考虑具有非规范幂型非线性项和混合分数阶算子的如下耦合薛定谔方程组:

$$\begin{cases} i\partial_t u + (-\triangle)^{\frac{\alpha}{2}} u = \lambda_1 |u|^p + \lambda_2 |u||v|^{p-1} & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ i\partial_t v + (-\triangle)^{\frac{\beta}{2}} v = \lambda_2 |u|^{p-1}|v| + \lambda_3 |v|^p & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ (u,v)(x,0) = (u_0, v_0) \end{cases} \tag{1}$$

这里$\alpha, \beta \in (0,2]$.λ_1, λ_2和$\lambda_3 \in \mathbb{C} \setminus \{0\}$,$(-\triangle)^{\frac{\alpha}{2}} := \mathscr{F}^{-1}|\xi|^\alpha \mathscr{F}$,此处$\mathscr{F}$表示傅里叶变换, 定义为:

$$(\mathscr{F}f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\xi} f(x) dx.$$

这个系统源自于控制双折射光纤中两个偏振分量的如下规范型非线性薛定谔方程组(见 [7]):

$$\begin{cases} i\partial_t u + (-\triangle)^{\frac{\alpha}{2}} u = (a|u|^{p-1} + b|v|^{p-1})u \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ i\partial_t v + (-\triangle)^{\frac{\beta}{2}} v = (b|u|^{p-1} + c|v|^{p-1})v \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ (u,v)(x,0) = (u_0, v_0) \end{cases} \tag{2}$$

其中$a,b,c \in \mathbb{R}, u,v : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}^N, N \geq 1$. 关于(2), 如果令$u(x) = e^{i\omega_1 t}u_1(x), v(x) = e^{i\omega_2 t}v_1(x)$, 它将转换为一个耦合的椭圆系统, 因此受到很多学者的关注. 例如当$\alpha = \beta = 2$ 时, Bartsch和Wang [8] 研究了(2) 在N 维情形下($N \leq 3$) 的正基态解存在和不存在的充分必要条件. Cipolatti,Zumpichiatti [9]研究了一维NLSEs驻波的存在性和稳定性(特别是所谓的基态解). Lopes [10, 11]研究了NLSEs的孤立波的稳定性. Nguyen, Wang [12] 证明了(2) 在$N = 1, p = 2$ 时孤立波的存在性和稳定性与a,b,c 的关系. Cely, Goloshchapova [13]首次研究了当$p = 3$ 时, 方程组(2) 在星图上的变分性和稳定性. 此外, Chen等 [14]指出一维情形下方程组(2)会像δ 函数一样爆破. 进一步 [15]中研究了在$N \leq 2, p = 3, \alpha = 2$ 时, 解的爆破二择性. 最近在文献 [16]中作者推广该结论至三维情形, 证明了能量$E(u_0, v_0) < 0$ 时, 方程组解(u,v) 的爆破特性.

在玻色-爱因斯坦凝聚中, 通常需要考虑解在空间无穷远处与非零常数的接近程度. 在这种情况下,如果我们通过$u = u' + C$(C 为常数)引入新的动力变量u', 并在方程中展开, 就会出现非线性项$|u'|^p$([17,18]), 而这个项的出现在很大程度上会影响方程组解的性态研究. 正如Fujiwara 和Ozawa 在文献 [19]中对单个薛定谔方程指出的那样, 在一维空间中当$\alpha = 1$ 时, 如果非线性项为$|u|^{p-1}u$, 方程将在$p \in (1,3)$时存在全局解, 而若非线性项变为$|u|^p$, 解将在$p \in (1,2]$ 的条件下爆破.

相较于(2), 方程组(1)不再拥有质能守恒属性, 因此前述关于(2)的研究方法和结论对(1)并不适用. 本文中我们聚焦于系统(1) 的爆破属性, 通过引入测试函数, 结合矛盾论证证明(1)会在有限时间内爆破. 这里的关键在于选择一个合适的测试函数来克服混合分数阶算子和耦合的非线性项带来的困难. 除此之外, 我们还得到了爆破解存在时间的上界估计.

文章中, 将$f \leq Cg$ 写作$f \lesssim g$, 将$f \lesssim g \lesssim f$写作$f \sim g$, 其中C是一个正常数, 且文章中每一个C的值可能都会不同.下面, 我们来介绍本文的主要结论.

定理1.1. 令$u_0, v_0 \in L^2(\mathbb{R}^n)$, 满足

$$M_R(0) + (L_R(0) - 2C_{n,p,\rho,\lambda}R^{n-\gamma/p} > 0,$$

其中$\lambda = \min\{|\Re(\lambda_1 + \lambda_2)|, \Re(\lambda_3 + \lambda_2)|\}, R > 0, \rho \in \mathbb{C}$ 并且当$\Re(\lambda_1\lambda_2) > 0, \Re(\lambda_3\lambda_2) > 0$ 满足

$$\Re(\rho\lambda_1 > 0); \Re(\rho\lambda_3 > 0),$$

而当$\Re(\lambda_1\lambda_2) < 0, \Re(\lambda_3\lambda_2) < 0$时,需要进一步满足

$$\Re(\rho(\lambda_1 + \lambda_2 > 0), \Re(\rho(\lambda_3 + \lambda_2) > 0,$$

其中$M_R(0), L_R(0)$与$C_{n,p,\rho,\lambda}$定义如下

$$M_R(0) = -\Im(\rho \int_{\mathbb{R}^n} u_0(x)\psi_R(x)dx); \quad L_R(0) = -\Im(\rho \int_{\mathbb{R}^n} v_0(x)\psi_R(x)dx),$$

$$C^{p}_{n,p,\rho,\lambda} = 2^{1+q/p}p^{-q/p}q^{-1}\Re(\rho(\lambda_1+\lambda_2))^{-q/p}A^{q}_{n,\gamma}(\int \psi(x)dx)^{p};$$

则当初值$(u(0),v(0))=(u_0,v_0)$且$T>T_{n,p,\lambda,\alpha,\beta}$ 时, (1)在空间$X(T)$中无解, 其中

$$T_{n,p,\lambda,\alpha,\beta} = (M_R(0)+L_R(0)-2C_{n,p,\rho,\lambda}R^{n-\gamma/p})^{1-p}$$

$$\cdot((p-1)D_{n,p,\lambda,\rho}R^{-n(p-1)})^{-1},$$

$$D_{n,p,\lambda,\rho} = 2^{-1}\Re(\rho(\lambda)|\rho|^{-p}(\int_{\mathbb{R}^n}\psi(x)dx)^{-p+1}(\sum_{j=0}^{p}\frac{p!}{j!(p-j-1)!})^{-1},$$

$$X(T)=C([0,T);L^2(\mathbb{R}^n))\cap C^1([0,T),H^{-\beta}(\mathbb{R}^n))\cap C^1([0,T),H^{-\alpha}(\mathbb{R}^n))\cap L^\infty(0,T;L^p(\mathbb{R}^n)).$$

2. 证明所需定义与引理

定义2.1. [20] 令$\alpha\in(0,2)$. 令X是一个定义在$\mathbb{R}^N$上的函数集合. 则$\mathbb{R}^N$中的分数阶Laplace算子$(-\triangle)^{\frac{\alpha}{2}}$是一个非局部算子, 定义如下:

$$(-\triangle)^{\frac{\alpha}{2}}: v\in X\to(-\triangle)^{\frac{\alpha}{2}}v := C_{N,\alpha}P.V\int_{\mathbb{R}^N}\frac{v(x)-v(y)}{|x-y|^{N+\alpha}}dy \tag{3}$$

$P.V.$表示柯西主值, $C_{N,\alpha}$表示依赖N,α的常数.

定义2.2. 如果$u,v\in L^1_{loc}(0,T;L^2(\mathbb{R}^N))\cap L^1_{loc}(0,T;L^p(\mathbb{R}^N))$ 且下面的等式

$$\begin{cases}\int_0^T(u,i\partial_t\varphi+(-\triangle)^{\frac{\alpha}{2}}\varphi)dt=\int_0^T(\lambda_1|u|^p+\lambda_2|u||v|^{p-1},\varphi)dt+i(u_0,\varphi_0) & (4)\\ \int_0^T(v,i\partial_t\varphi+(-\triangle)^{\frac{\beta}{2}}\varphi)dt=\int_0^T(\lambda_2|u|^{p-1}|v|+\lambda_3|v|^p,\varphi)dt+i(v_0,\varphi_0) & (5)\end{cases}$$

对任意的测试函数$\varphi\in C([0,T];H^m(\mathbb{R}^N))\cap C^1([0,T];L^2(\mathbb{R}^N))$,$m\geq\max\{\alpha,\beta\}$,$\varphi(T,x)=0$ 在$x\in\mathbb{R}^N$上成立, 我们称(u,v)是方程组(1)的弱解. 这里的$(\cdot,\cdot)$是L^2的内积, 定义为$(f,g)=\int_{\mathbb{R}^N}f(x)\overline{g(x)}$.

引理2.1. 令$\langle x\rangle=\sqrt[3]{1+(|x|-1)^3}$, 且$\gamma=\min\{\alpha,\beta\}$, 令$\psi(x)$ 是一个连续分段函数, 定义如下:

$$\psi(x)=\begin{cases}1 & |x|\leq1\\ \langle x\rangle^{-N-\gamma} & |x|\geq1\end{cases} \tag{6}$$

则对任意的$x\in\mathbb{R}^N,\psi\in C^2(\mathbb{R}^N)$, 存在一个仅依赖于$N,\alpha$的正常数$A_{N,\alpha}$ 使得$(-\triangle)^{\frac{\alpha}{2}}\psi$满足以下逐点估计

$$|(-\triangle)^{\frac{\alpha}{2}}\psi|\leq A_{N,\alpha}\psi(x) \tag{7}$$

证: 为了方便, 我们令$r := |x|$, 并考虑$\psi(x)$的导数:

$$\nabla\psi(x) = \frac{x}{r}\psi'(x) = \begin{cases} 0, & r \leq 1 \\ -(N+\gamma)(r-1)^2 \frac{x}{r}\langle x\rangle^{-N-\gamma-3}, & r \geq 1 \end{cases} \tag{8}$$

和

$$\triangle\psi(x) = \frac{N-1}{r}\psi'(x) + \psi''(x) = \begin{cases} 0, & r \leq 1 \\ -(N+\gamma)(r-1)^2 \frac{N-1}{r}\langle x\rangle^{-N-\gamma-3} & \\ -2(N+\gamma)(r-1)\langle x\rangle^{-N-\gamma-3} & \\ +(N+\gamma)(N+\gamma+3)(r-1)^4\langle x\rangle^{-N-\gamma-6}, & r \geq 1 \end{cases} \tag{9}$$

容易验证$\psi(x) \in C^2(\mathbb{R}^N)$且$|\triangle\psi(x)| \leq \langle x\rangle^{-N-\alpha}$.因此, 我们只需要考虑$0 < \alpha < 2$. 除此之外, 通过上述计算, 我们可以得到$\|\psi\|, \|\partial_x^2\psi\|_{L^\infty} < \infty$, 这可以使我们去掉积分在原点的主值. 根据定义(2.1) 并利用变量替换可得:

$$\begin{aligned}(-\triangle)^{\frac{\alpha}{2}}\psi(x) &= -\frac{C_{N,\alpha}}{2} P.V. \int_{\mathbb{R}^N} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy \\ &= -\frac{C_{N,\alpha}}{2} P.V. \lim_{\sigma\to 0+} \int_{\sigma\leq|y|\leq 1} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy \\ &= -\frac{C_{N,\alpha}}{2} P.V. \int_{|y|\geq 1} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy \end{aligned} \tag{10}$$

对ψ做二阶泰勒展开可得:

$$\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} \lesssim \frac{\|\partial_x^2\psi\|_{L^\infty}}{|y|^{N+\alpha-2}} \tag{11}$$

通过上述估计和$\alpha \in (0, 2)$, 我们可以通过去除积分在原点的主值来得到如下结论:

$$(-\triangle)^{\frac{\alpha}{2}}\psi(x) = -\frac{C_{N,\alpha}}{2} P.V. \int_{\mathbb{R}^N} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy \tag{12}$$

为了得到我们期待的结果, 下面我们分别讨论$|x| \leq 2$和$|x| \geq 2$的情形. 对于$|x| \leq 2$, 我们将积分区域分为两部分:

$$\Gamma_1 = \{(x,y) : |y| \leq 1\}, \Gamma_2 = \{(x,y) : |y| \geq 1\}.$$

在Γ_1和Γ_2上, 分别有如下估计:

$$\int_{\Gamma_1} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy \lesssim \|\partial_x^2\psi\|_{L^\infty} \int_{\Gamma_1} \frac{1}{|y|^{N+\alpha-2}} \lesssim 1 \tag{13}$$

$$\int_{\Gamma_2} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy \lesssim \|\psi\|_{L^\infty} \int_{\Gamma_2} \frac{1}{|y|^{N+\alpha}} \lesssim 1 \tag{14}$$

对于$|x| \geq 2$, 我们将积分区域分成三部分:

$$\Gamma_3 = \{(x,y) : |y| \geq 2|x|\}, \Gamma_4 = \{(x,y) : \frac{1}{2}|x| \leq |y| \leq 2|x|\}, \Gamma_5 = \{(x,y) : |y \leq \frac{1}{2}|x|\}.$$

事实上, 当$|x| \geq 2$时, 有$\langle x \rangle \sim |x| - 1 \sim |x|$.

在Γ_3上, 我们发现$|x \pm y| \geq |y| - |x| \geq |x| \geq 2$.由$\psi$的单调性可得: $\psi(x \pm y) \leq \psi(x) = \langle x \rangle^{-N-\gamma}$.由此, 我们可以得出如下估计:

$$\begin{aligned}\int_{\Gamma_3} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy &\lesssim 4\psi(x) \int_{|y| \geq 2|x|} \frac{1}{|y|^{N+\alpha}} dy \\ &\lesssim \langle x \rangle^{-N-\gamma} \int_{|y| \geq 2|x|} \frac{1}{|y|^{1+\alpha}} d|y| \\ &\lesssim \langle x \rangle^{-N-\gamma} |x|^{-\alpha} \\ &\lesssim \langle x \rangle^{-N-\gamma-\alpha}\end{aligned} \tag{15}$$

在Γ_4上, 由$|y| \sim |x|$和$\Gamma_4 \in \{y \in \mathbb{R}^N : |x \pm y| \geq 2\}$可得:

$$\begin{aligned}&\int_{\Gamma_4} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy \\ &\lesssim |x|^{-N-\alpha} (\int_{|x+y| \leq 3|x|} \psi(x+y) dy + \int_{|x-y| \leq 3|x|} \psi(x-y) dy + 2\psi(x) \int_{\frac{1}{2}|x| \leq |y| \leq 2|x|} 1 dy) \\ &\lesssim \langle x \rangle^{-N-\alpha} (\int_{|x+y| \leq 3|x|} \psi(x+y) dy + \langle x \rangle^{-N-\gamma} |x|^N) \\ &\lesssim \langle x \rangle^{-N-\alpha} (|x|^{-\gamma} + 1) \\ &\lesssim \langle x \rangle^{-N-\gamma}\end{aligned} \tag{16}$$

这里我们使用了如下估计:

$$\begin{aligned}\int_{|x-y| \leq 3|x|} \psi(x-y) dy &= \int_{|x+y| \leq 3|x|} \psi(x+y) dy \\ &= \int_{2 \leq |x+y| \leq 3|x|} \psi(x+y) dy + \int_{|x+y| \leq 2} \psi(x+y) dy \\ &\lesssim \int_2^{3|x|} (1 + (r-1)^3)^{-(N+\gamma)/3} r^{N-1} dr + \int_0^2 r^{N-1} dr \\ &\lesssim \int_2^{3|x|} r^{-\gamma-1} dr + \int_0^2 r^{N-1} dr \\ &\lesssim 1\end{aligned} \tag{17}$$

接下来我们处理Γ_5上的积分. 利用ψ的二阶泰勒展开可得:

$$\int_{\Gamma_5} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{N+\alpha}} dy \lesssim \int_{\Gamma_5} \max_{\delta \in [0,1]} \partial_x^2 \psi(x \pm \delta y) \frac{1}{|y|^{N+\alpha-2}} dy \tag{18}$$

下面我们考虑关于$\partial_x^2\psi(x\pm\delta y)$的估计, 令$r=|x+\delta y|,\delta\in[0,1]$, 有:

$$
\begin{aligned}
|\partial_x^2\psi(x\pm\delta y)| &\lesssim \begin{cases} 0, & r\le 1,\\ \frac{(r-1)^2}{r}\langle r\rangle^{-N-\gamma-3}-(r-1)\langle r\rangle^{-N-\gamma-3}+(r-1)^4\langle r\rangle^{-N-\gamma-6}, & r\ge 1.\end{cases}\\
&\lesssim \begin{cases} 0, & r\le 1,\\ \langle r\rangle^{-N-\gamma-2}, & r\ge 1.\end{cases}
\end{aligned} \tag{19}
$$

由$|x\pm\delta y|\ge|x|-\delta|y|\ge|x|-|x|/2\ge 1$和上述估计可得:

$$|\partial_x^2\psi(x\pm\delta y)|\lesssim\langle x\pm\delta y\rangle^{-N-\gamma-2}.$$

如果$|x\pm\delta y|\ge 2$, 则$\langle x\pm\delta y\rangle\sim|x\pm\delta y|\ge|x|/2\gtrsim\langle x\rangle$. 由此可得:

$$\langle x\pm\delta y\rangle^{-N-\gamma-2}\lesssim\langle x\rangle^{-N-\gamma-2}$$

如果$|x\pm\delta y|\le 2$, 则$|x|/2\le|x\pm\delta|y|\le 2$, 这说明$2\le|x|\le 4$. 因此我们可以得到$\langle x\pm\delta y\rangle\sim|x|\sim 1$和$\langle x\pm\delta y\rangle^{-N-\gamma-2}\sim\langle x\rangle^{-N-\gamma-2}$, 则下式对所有的$\delta\in[0,1]$ 都成立

$$|\partial_x^2\psi(x\pm\delta y)|\lesssim\langle x\rangle^{-N-\gamma-2}.$$

由(18)可得:

$$
\begin{aligned}
\int_{\Gamma_5}\frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{N+\alpha}}dy &\lesssim\langle x\rangle^{-N-\alpha-2}\int_{\Gamma_5}\frac{1}{|y|^{N+\alpha-2}}dy\\
&\lesssim\langle x\rangle^{-N-\gamma-2}\int_0^{|x|/2}\frac{1}{|y|^{\alpha-1}}d|y|\\
&\lesssim\langle x\rangle^{-N-\gamma-2}|x|^{-\alpha+2}\\
&\lesssim\langle x\rangle^{-N-\gamma-\alpha}.
\end{aligned} \tag{20}
$$

结合(13)-(16)和(20)可得下式对所有的$x\in\mathbb{R}^N$都成立.

$$|(-\triangle)^{\frac{\alpha}{2}}\psi|\le\langle x\rangle^{-N-\gamma} \tag{21}$$

证毕. □

引理2.2. [21] 令$0<\alpha\le 2,x\in(\mathbb{R}^N),\psi$ 是一个满足$\partial_x^2\psi\in L^\infty(\mathbb{R}^N)$的光滑函数. 对任意的$R>0$, 令$\psi_R(x)=\psi(x/R)$. 则对所有的$x\in\mathbb{R}^N,(-\triangle)^{\frac{\alpha}{2}}\psi_R(x)$ 有如下性质:

$$(-\triangle)^{\frac{\alpha}{2}}\psi_R(x)=R^{-\alpha}((-\triangle)^{\frac{\alpha}{2}}\psi)(x/R)$$

3. 定理1.1的证明

设系统(1)存在一组解$(u,v)\in X(T)$, 其中$T>T_n,p,\lambda,\rho$. 令

$$M_R(T)=-\Im(\rho\int_{\mathbb{R}^N}u(x)\psi_R(x)dx);\quad L_R(T)=-\Im(\rho\int_{\mathbb{R}^N}v(x)\psi_R(x)dx),$$

则由引理2.1和引理2.2可得

$$\begin{aligned}\frac{d}{dt}M_R(t)&=\Re(\rho\int_{\mathbb{R}^N}i\partial_t u(t,x)\psi_R(x)dx)\\&=\Re(\int_{\mathbb{R}^N}(\rho\lambda_1|u(t,x)|^p+\rho\lambda_2|u||v|^{p-1})\psi_R(x)dx\\&-R^{-\alpha}\Re(\rho\int_{\mathbb{R}^N}u(t,x)((-\Delta)^{\alpha/2}\psi_R(x)dx)\\&\geq\Re\int_{\mathbb{R}^N}(\rho\lambda_1|u(t,x)|^p+\rho\lambda_2|u||v|^{p-1})\psi_R(x)dx\\&-A_{N,\gamma}R^{-\alpha}|\rho|\int_{\mathbb{R}^N}|u(t,x)|\psi_R(x)dx.\end{aligned}\tag{22}$$

同理可得

$$\begin{aligned}\frac{d}{dt}L_R(t)&\geq\Re\int_{\mathbb{R}^N}(\rho\lambda_1|v(t,x)|^p+\rho\lambda_2|u||v|^{p-1})\psi_R(x)dx\\&-A_{N,\gamma}R^{-\beta}|\rho|\int_{\mathbb{R}^N}|v(t,x)|\psi_R(x)dx.\end{aligned}\tag{23}$$

当$\Re(\lambda_1\lambda_2)<0,\Re(\lambda_3\lambda_2)<0$时, 利用Young不等式并结合(22)和(23)可得

$$\begin{aligned}\frac{d}{dt}L_R(t)+\frac{d}{dt}M_R(t)\geq&\Re\int_{\mathbb{R}^N}(\rho(\lambda_1+\lambda_2)|u(t,x)|^p\psi_R(x)dx\\&+\Re\int_{\mathbb{R}^N}(\rho(\lambda_3+\lambda_2)|v(t,x)|^p\psi_R(x)dx\\&-A_{N,\gamma}R^{-\beta}|\rho|\int_{\mathbb{R}^N}|v(t,x)|\psi_R(x)dx\\&-A_{N,\gamma}R^{-\alpha}|\rho|\int_{\mathbb{R}^N}|u(t,x)|\psi_R(x)dx.\end{aligned}\tag{24}$$

使用霍尔德不等式和Young不等式可得

$$
\begin{aligned}
&A_{N,\gamma}R^{-\alpha}|\rho|\int_{\mathbb{R}^N}|u(t,x)|\psi_R(x)dx\\
&\leq A_{N,\gamma}|\rho|R^{-\alpha}\left(\int_{\mathbb{R}^N}\psi_R(x)dx\right)^{1/q}\left(\int_{\mathbb{R}^N}|u(t,x)|^p\psi_R(x)dx\right)^{1/p}\\
&\leq A_{N,\gamma}|\rho|R^{N/q-\alpha}\left(\int_{\mathbb{R}^N}\psi(x)dx\right)^{1/q}\left(\int_{\mathbb{R}^N}|u(t,x)|^p\psi_R(x)dx\right)^{1/p}\\
&\leq p^{-q/p}q^{-1}2^{q/p}\Re((\rho(\lambda_1+\lambda_2)))^{-q/p}|\rho|^qA_{N,\gamma}^qR^{N-q\alpha}\int_{\mathbb{R}^N}\psi_(x)dx\\
&+2^{-1}\Re(\rho(\lambda_1+\lambda_2))\int_{\mathbb{R}^N}|u(t,x)|^p\psi_R(x)dx,
\end{aligned}
\tag{25}
$$

和

$$
\begin{aligned}
|M_R(t)|&=\left|\Im\left(\rho\int_{\mathbb{R}^N}u(t,x)\psi_R(x)dx\right)\right|\\
&\leq|\rho|R^{N/q}\left(\int_{\mathbb{R}^N}\psi(x)dx\right)^{1/q}\left(\int_{\mathbb{R}^N}|u(t,x)|^p\psi_R(x)dx\right)^{1/p}.
\end{aligned}
\tag{26}
$$

用相同的方法可得

$$
\begin{aligned}
&A_{N,\gamma}R^{-\beta}|\rho|\int_{\mathbb{R}^N}|v(t,x)|\psi_R(x)dx\\
&\leq p^{-q/p}q^{-1}2^{q/p}\Re((\rho(\lambda_3+\lambda_2)))^{-q/p}|\rho|^qA_{N,\gamma}^qR^{N-q\beta}\int_{\mathbb{R}^N}\psi(x)dx\\
&+2^{-1}\Re(\rho(\lambda_3+\lambda_2))\int_{\mathbb{R}^N}|v(t,x)|^p\psi_R(x)dx,
\end{aligned}
\tag{27}
$$

和

$$
\begin{aligned}
|L_R(t)|&=\left|\Im\left(\rho\int_{\mathbb{R}^N}v(t,x)\psi_R(x)dx\right)\right|\\
&\leq|\rho|R^{N/q}\left(\int_{\mathbb{R}^N}\psi(x)dx\right)^{1/q}\left(\int_{\mathbb{R}^N}|v(t,x)|^p\psi_R(x)dx\right)^{1/p}.
\end{aligned}
\tag{28}
$$

联立(24)-(28),并使用如下不等式

$$
(a+b)^p\leq\sum_{j=0}^{p}\frac{(p-1)!}{j!(p-j-1)!}(a^p+b^p),
$$

我们得到

$$
\begin{aligned}
&\frac{d}{dt}L_R(t)+\frac{d}{dt}M_R(t)\\
&\geq 2^{-1}\Re(\rho(\lambda_1+\lambda_2))|\rho|^{-p}\left(\int_{\mathbb{R}^N}\psi(x)dx\right)^{-p+1}R^{-N(p-1)}M_R(t)^p\\
&+2^{-1}\Re(\rho(\lambda_3+\lambda_2))|\rho|^{-p}\left(\int_{\mathbb{R}^N}\psi(x)dx\right)^{-p+1}R^{-N(p-1)}L_R(t)^p\\
&-p^{-q/p}q^{-1}2^{q/p}\Re((\rho(\lambda_1+\lambda_2)))^{-q/p}|\rho|^qA^q_{N,\gamma}R^{N-q\alpha}\int_{\mathbb{R}^N}\psi_(x)dx\\
&-p^{-q/p}q^{-1}2^{q/p}\Re((\rho(\lambda_3+\lambda_2)))^{-q/p}|\rho|^qA^q_{N,\gamma}R^{N-q\beta}\int_{\mathbb{R}^N}\psi(x)dx\\
&\geq 2^{-1}\Re(\rho(\lambda)|\rho|^{-p}R^{-N(p-1)}\left(\int_{\mathbb{R}^N}\psi(x)dx\right)^{-p+1}\\
&\cdot((M_R(t)^p-C^p_{N,p,\rho,\lambda}R^{Np-q\alpha})+(L_R(t)^p-C^p_{N,p,\rho,\lambda}R^{Np-q\beta}))\\
&\geq 2^{-1}\Re(\rho(\lambda)|\rho|^{-p}R^{-N(p-1)}\left(\int_{\mathbb{R}^N}\psi(x)dx\right)^{-p+1}R^{-N(p-1)}\\
&\cdot((M_R(t)-C_{N,p,\rho,\lambda}R^{N-\alpha/(p-1)})^p+(L_R(t)-C_{N,p,\rho,\lambda}R^{N-\beta/(p-1)})^p)
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
&\geq D_{N,p,\lambda,\rho}R^{-N(p-1)}((M_R(t)-C_{N,p,\rho,\lambda}R^{N-\alpha/(p-1)})\\
&+(L_R(t)-C_{N,p,\rho,\lambda}R^{N-\beta/(p-1)}))^p.
\end{aligned}
\tag{30}
$$

通过解常微分方程可得

$$
\begin{aligned}
&\frac{d}{dt}(L_R(t)+M_R(t))\\
&\geq\{((M_R(0)-C_{N,p,\rho,\lambda}R^{N-\alpha/(p-1)})+(L_R(0)-C_{N,p,\rho,\lambda}R^{N-\beta/(p-1)}))^{1-p}\\
&-(p-1)D_{N,p,\lambda,\rho}R^{-N(p-1)}t\}^{-1/(p-1)}\\
&>0,
\end{aligned}
\tag{31}
$$

则(31)式右边会在$t=T_{N,p,\lambda,\alpha,\beta}$ 处爆破，$L_R(t),M_R(t)$也随之爆破. 因为

$$M_R(t)\leq\|u\|_{L^2(\mathbb{R}^N)}\|\psi_R(x)\|_{L^2(\mathbb{R}^N)};$$

$$L_R(t)\leq\|v\|_{L^2(\mathbb{R}^N)}\|\psi_R(x)\|_{L^2(\mathbb{R}^N)},$$

则(31)式与(1)在$C(0,T;L^2(\mathbb{R}^N)),T>T_{N,p,\lambda,\alpha,\beta}$ 空间中有解相矛盾.

当$\Re(\lambda_1\lambda_2) > 0, \Re(\lambda_3\lambda_2) > 0$时, 我们可以使用相同的方法来得到相似得结果, 所以我们省略其证明过程. 证毕.

4. 总结

本文通过引入测试函数, 推导出关于解的加权积分的常微分不等式, 利用常微分方程的性质证明了解会在有限时间内爆破. 这一方法可以直观的展示解的爆破行为, 与经典的耦合薛定谔方程组(2) 相比, 本文引入了混合分数阶算子, 且非线性也发生改变, 这使得本文研究的方程组不再具备质能守恒属性, 研究方法也有了相应的改变. 尽管本文已通过测试函数法有效地探究了方程组解的爆破特性和存在时间估计, 但对爆破解并没有做更深入的研究. 未来的研究方向将聚焦于揭示解的爆破速率. 并利用数值模拟, 来更清晰地展示解的爆破行为.

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